Polynomial knot invariants via R-matrices

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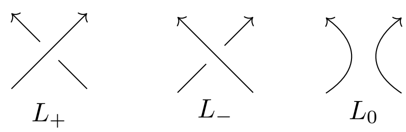
In this document we present a construction of the Jones polynomial, denoted $V_L(\cdot) \in \mathbb{Z}[t^{-1/2}, t^{1/2}]$, using R-matrices. The Jones polynomial is uniquely characterized by the two following relations:

$$t^{-1}V_{L^+}(t) - tV_{L^-}(t) = (t^{1/2} - t^{-1/2})V_{L_0}(t) \quad (\textit{skein relation}) \tag{1}$$
$$V_O(t) = 1 \tag{2}$$
$$\tag{3}$$

where O denotes the trivial knot and,



1 Braid group

Definition 1.1. The *braid group in n strands*, denoted $\mathcal{B}_n$, is defined by $n-1$ generators $\sigma_1, \dots, \sigma_{n-1}$ and the relations:

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad |i-j| \geq 2, \tag{4}$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad i \in \{1, \dots, n-1\}. \tag{5}$$

There is a unique surjective group homomorphism from $\mathcal{B}_n$ to $\mathfrak{S}_n$ that send σ_i to $\tau_i = (i \quad i+1)$.

2 Topology of braids

Theorem 2.1. Two braid diagrams represent the same braid if and only if they are related by a finite sequence of braid moves. That is:

$$\frac{\{\text{braids}\}}{\text{equivalence}} \cong \frac{\{\text{braid diagrams}\}}{\text{braid moves}}.$$

Remark 2.2.
$$\frac{\{\text{braids}\}}{\text{equivalence}} \ncong \frac{\{\text{oriented links}\}}{\text{equivalence}}.$$

Theorem 2.3 (Alexander's theorem)**.** Every oriented link arises as the closure of a braid.

Theorem 2.4. The closure of two braid diagrams σ and σ' represent equivalent links if and only if there is a finite sequence of braid moves and Markov moves taking σ to σ'. That is

$$\widetilde{\text{Link}} := \frac{\{\text{oriented links}\}}{\text{equivalence}} \cong \frac{\{\text{braids}\}}{\text{braid moves, Markov moves}}.$$

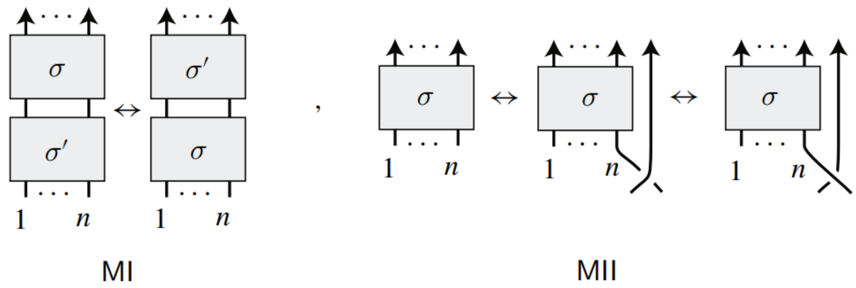


Figure 1: Markov moves

3 Representation theory

3.1 $\mathcal{B}_n$ representations with R-matrices

Let V be a two dimensional vector space (or a free rank two $\mathbb{Q}[t^{1/2}, t^{-1/2}]$-module). We denote by $\tau_i = (i\ i+1) \in \mathfrak{S}_n$ the transposition between i and $i+1$. These elements generate $\mathfrak{S}_n$ as a group. One can define the following representation of the symmetric group:

$$\phi_n : \left| \begin{array}{ccc} \mathfrak{S}_n & \longrightarrow & V^{\otimes n} \\ \tau_i & \longmapsto & \mathrm{Id}^{\otimes(i-1)} \otimes P \otimes \mathrm{Id}^{\otimes(n-i-1)} \end{array} \right. .$$

where

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{End}(V \otimes V).$$

Then, thanks to the natural surjective homomorphism $\mathcal{B}_n \to \mathfrak{S}_n$ defined by $\sigma_i \mapsto \tau_i$, we have a representation of the braid group:

$$\psi_n : \left| \begin{array}{ccc} \mathcal{B}_n & \longrightarrow & V^{\otimes n} \\ \sigma_i & \longmapsto & \mathrm{Id}^{\otimes(i-1)} \otimes P \otimes \mathrm{Id}^{\otimes(n-i-1)} \end{array} \right. .$$

Modifying the above representation we try to obtain a representation $\psi_n : \mathcal{B}_n \to \mathrm{End}(V^{\otimes n})$ defined by:

$$\psi_n(\sigma_i) = \mathrm{Id}^{\otimes(i-1)} \otimes R \otimes \mathrm{Id}^{\otimes(n-i-1)},$$

for some linear map $R : V \otimes V \to V \otimes V$. To obtain such a representation ψ_n of the braid group $\mathcal{B}_n$, the map ψ_n is required to satisfy the following relations,

$$\psi_n(\sigma_i\sigma_j) = \psi_n(\sigma_j\sigma_i), \qquad |i-j| \leq 2, \tag{6}$$
$$\psi(\sigma_i\sigma_{i+1}\sigma_i) = \psi(\sigma_{i+1}\sigma_i\sigma_{i+1}). \tag{7}$$

The relation (6) is always satisfied. To obtain (7), the matrix R must satisfy the relation:

$$(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R)(R \otimes \mathrm{Id}_V) = (\mathrm{Id}_V \otimes R)(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R).$$

This equation is called the *Yang-Baxter equation*, and a solution to it is called a *R-matrix*.

If we denote by $R(e_k \otimes e_l) = \sum_{i,j} R^{k,l}_{i,j} e_i \otimes e_j$, and $R^{k,l}_{i,j} = 0$ if $i + j = k + l$, we say that R satisfies the property of *charge conservation*. In this case, R preserves the three subspaces of $V \otimes V$ spanned by the bases $\{e_0 \otimes e_0\}$, $\{e_0 \otimes e_1, e_1 \otimes e_0\}$ and $\{e_1 \otimes e_1\}$. This allows us to use the following notation:

$$R = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & d & e & 0 \\ 0 & 0 & 0 & f \end{pmatrix}$$

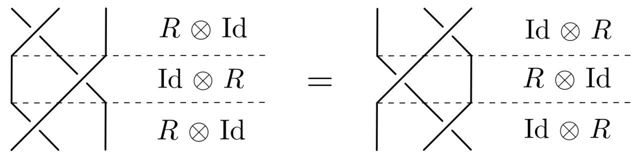


Figure 2: Illustration of Yang-Baxter's equation

with some computations, the Yang-Baxter equation turns into:

$$b(cd + ab + a^2) = 0, \quad b(cd + bf - f^2) = 0$$
$$e(cd + ae - a^2) = 0, \quad e(cd + ef - f^2) = 0$$
$$be(b - e) = 0, \quad bde = 0, \quad bce = 0.$$

These equations are symmetrics under the interchange of b and e. If one supposes that b and e are not equal to 0, then the equations imply that $b = e$ and $a = c = d = f = 0$. This does not define a representation. Therefore, we consider the case that $b = 0$ and $e \neq 0$. In this case,

$$a^2 - ae = cd = f^2 - ef.$$

Hence,

$$e = a - \frac{cd}{f}, \qquad (a - f)(a + f - e) = 0.$$

Corresponding to the two cases $a - f = 0$ and $a + f - e = 0$, we obtain the following two R-matrices:

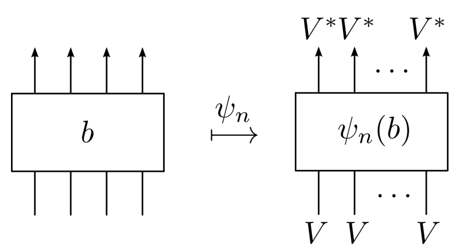
$$R_1 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & d & a - cd/a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \qquad R_2 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & d & a - cd/a & 0 \\ 0 & 0 & 0 & -cd/a \end{pmatrix}. \tag{8}$$

3.2 Markov moves and representations.

Let V be a finite-dimensional vector space over $\mathbb{C}$ (or a free rank-two $\mathbb{Q}[t^{-1/2}, t^{1/2}]$-module) with basis $B = \{e_i\}_i$. We have the following identification:

$$u : \left| \begin{array}{ccc} \mathrm{End}(V) & \xrightarrow{\sim} & V^* \otimes V \\ f = \sum_{i=1}^n f_i e_i & \longmapsto & \sum_{i=1}^n f_i \otimes e_i \end{array} \right.$$

This identification leads to the following diagrammatic presentation of a braid group representation:



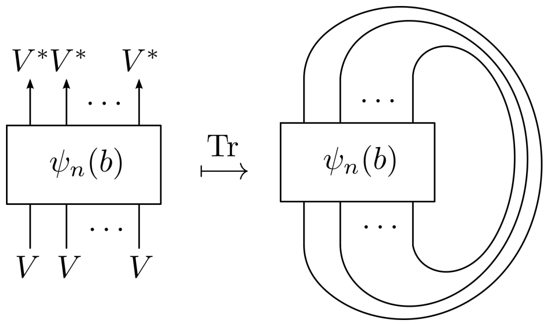
The idea of finding a link invariant is now to mimic the closure of a braid. Thanks to the previous presentation of a braid representation, we can now think of evaluating each linear form on each element of V. To do this, we will use the trace of a linear map. This can be seen as follows:

$$\mathrm{Tr} : \mathrm{End}(V) \xrightarrow{u} V^* \otimes V \xrightarrow{v} \mathbb{C},$$

where,

$$v : \left| \begin{array}{ccc} V^* \otimes V & \longrightarrow & \mathbb{C} \\ f \otimes x & \longmapsto & f(x) \end{array} \right. .$$

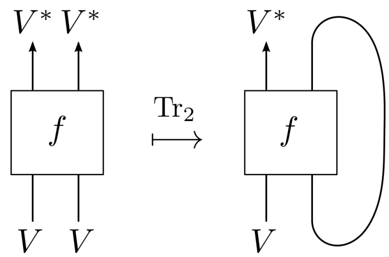
Diagrammatically, we will represent it as follows:



Further, we can define:

$$\mathrm{Tr}_2 : \left| \begin{array}{ccc} \mathrm{End}(V \otimes V) \simeq V^* \otimes V^* \otimes V \otimes V & \longrightarrow & V^* \otimes V \simeq \mathrm{End}(V) \\ g_j \otimes f_i \otimes e_i \otimes b_j & \longmapsto & g_j(b_j) f_i \otimes e_i \end{array} \right. .$$

Diagrammatically, this can be represented as:



For example if V is a two dimensional vector space with basis $\{e_0, e_1\}$, a linear map $A \in \mathrm{End}(V \otimes V)$ is represented by a matrix $A = (a_{i,j})_{1 \leq i,j \leq 4}$, with respect to the basis $\{e_0 \otimes e_0, e_0 \otimes e_1, e_1 \otimes e_0, e_1 \otimes e_1\}$ of $V \otimes V$. Then

$$\mathrm{Tr}_2(A) = \begin{pmatrix} a_{1,1} + a_{2,2} & a_{1,3} + a_{2,4} \\ a_{3,1} + a_{4,2} & a_{3,3} + a_{4,4} \end{pmatrix}.$$

We remark that, in the case of a two dimensional vector space,

$$\mathrm{Tr}(\mathrm{Tr}_2(A)) = \mathrm{Tr}(A)$$

that is consistent with the diagrammatical presentation.

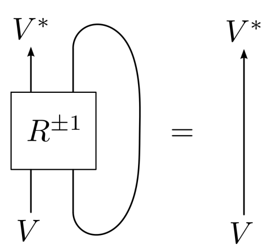
Proposition 3.1. Let $(\psi_n)_n$ be a family of representations of the groups $\mathcal{B}_n$ associated with the R-matrix R. One defines

$$\rho_R : \left| \begin{array}{ccc} \widetilde{\mathrm{Link}} & \longrightarrow & \mathbb{C} \\ L & \longmapsto & \mathrm{Tr}(\psi(b_L)) \end{array} \right.$$

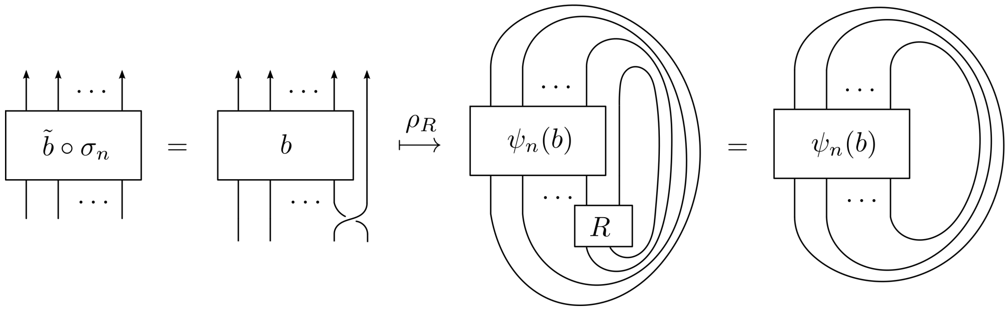
where b_L is a braid whose closure is isotopic to L. Then ρ_R is well-defined and a link invariant if and only if

$$\mathrm{Tr}_2(R^{\pm 1}) = \mathrm{Id}_V. \tag{9}$$

Diagrammatically, we can intuitively understand this proposition. Indeed, the fact that $\mathrm{Tr}_2(R^{\pm 1}) = \mathrm{Id}_V$ is presented as follows:



For a given braid $b \in \mathcal{B}_n$, we denote by $\tilde{b} := b \otimes 1 \in \mathcal{B}_{n+1}$, thus,



For the real proof of the proposition, we need to introduce:

$$\begin{aligned}\mathrm{Tr}_{n+1} : \mathrm{End}(V^{\otimes n+1}) &\simeq (V^{\otimes n} \otimes V)^* \otimes (V^{\otimes n} \otimes V) \\ &\simeq (V^{\otimes n})^* \otimes V^* \otimes V^{\otimes n} \otimes V \simeq (V^{\otimes n})^* \otimes V^* \otimes V \otimes V^{\otimes n} \\ &\xrightarrow{\mathrm{Id}^{\otimes n} \otimes v \otimes \mathrm{Id}^{\otimes n}} (V^{\otimes n})^* \otimes V^{\otimes n} \simeq \mathrm{End}(V^{\otimes n})\end{aligned}$$

Lemma 3.2. These family of linear maps satisfy the following identities, let $f \in \mathrm{End}(V^{\otimes n+1})$, $g \in \mathrm{End}(V^{\otimes n})$ and $h \in \mathrm{End}(V \otimes V)$:

1. $\mathrm{Tr}(\mathrm{Tr}_{n+1}(f)) = \mathrm{Tr}(f)$,
2. $\mathrm{Tr}_{n+1}(f \circ (g \otimes \mathrm{Id}_V)) = \mathrm{Tr}_{n+1}(f) \circ g$,
3. $\mathrm{Tr}_{n+1}((g \otimes \mathrm{Id}_V) \circ f) = g \circ \mathrm{Tr}_{n+1}(f)$,
4. $\mathrm{Tr}_{n+1}(\mathrm{Id}_V^{\otimes n-1} \otimes h) = \mathrm{Id}_V^{\otimes n-1} \otimes \mathrm{Tr}_2(h)$.

One can prove the Proposition 3.1:

Proof. Thanks to Theorem 2.4, we need to prove that $\mathrm{Tr}(\psi(\cdot))$ satisfies (9) iff $\mathrm{Tr}(\psi(\cdot))$ is invariant under Markov moves. The invariance under MI is follows from the properties of the trace, for all $b, b' \in \mathcal{B}_n$:

$$\mathrm{Tr}(\psi(bb')) = \mathrm{Tr}(\psi(b)\psi(b')) = \mathrm{Tr}(\psi(b')\psi(b)) = \mathrm{Tr}(\psi(b'b)),$$

so it is always true.

For MII, let us use Lemma 3.2,

$$\mathrm{Tr}(\psi((b \otimes 1)\sigma_n^{\pm 1})) = \mathrm{Tr}(\mathrm{Tr}_{n+1}(\psi(b \otimes 1) \circ \psi(\sigma_n^{\pm 1}))),$$

because $\psi(b \otimes 1) = \psi(b) \otimes \mathrm{Id}_V$, we have,

$$\mathrm{Tr}_{n+1}(\psi(b) \otimes \mathrm{Id}_V \circ \psi(\sigma_n^{\pm 1})) = \psi(b) \circ \mathrm{Tr}_{n+1}((\mathrm{Id}_V^{\otimes n-1} \otimes R^{\pm 1})) = \psi(b) \circ (\mathrm{Id}_V^{\otimes n-1} \otimes \mathrm{Tr}_2(R^{\pm 1}))$$

□

The problem is that, with the matrices R_1 and R_2 from (10), imposing $\mathrm{Tr}_2(R_i^{\pm 1}) = \mathrm{I}_2$ is too strong a condition, because this leads to:

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{d} & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & d & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{10}$$

The idea now is to take a matrix that satisfies one of the two forms in (10) and use the corresponding twisted representation to enforce the Markov moves.

3.3 Jones polynomial via R-matrices

We choose the matrix:

$$R_1 = \begin{pmatrix} -t^{1/2} & 0 & 0 & 0 \\ 0 & 0 & -t & 0 \\ 0 & -t & t^{3/2} - t^{1/2} & 0 \\ 0 & 0 & 0 & -t^{1/2} \end{pmatrix} \in \mathrm{End}(V \otimes V).$$

To create a link invariant thanks to the previous matrix, we consider the following linear map:

$$h = \begin{pmatrix} -t^{-1/2} & 0 \\ 0 & -t^{1/2} \end{pmatrix} \in \mathrm{End}(V).$$

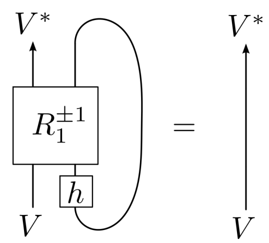
Then,

$$(\mathrm{Id}_V \otimes h) \cdot R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & t^{3/2} & 0 \\ 0 & t^{1/2} & 1 - t & 0 \\ 0 & 0 & 0 & t \end{pmatrix}$$

and,

$$(\mathrm{Id}_V \otimes h) \cdot R_1^{-1} = \begin{pmatrix} t^{-1} & 0 & 0 & 0 \\ 0 & 1 - t^{-1} & t^{-1/2} & 0 \\ 0 & t^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

therefore, $\mathrm{Tr}_2((\mathrm{Id}_V \otimes h) R_1^{\pm 1}) = \mathrm{Id}_V$. Schematically, this gives:



Furthermore, by the charge conservation property of R_1, it is straightforward to check that

$$(h \otimes h) \cdot R_1 = R_1 \cdot (h \otimes h).$$

Theorem 3.3. Let L be an oriented link and b a braid whose closure is isotopic to L. Then,

$$V_b^{\mathcal{B}}(t) = \mathrm{Tr}(\psi(b) \cdot h^{\otimes n}), \tag{11}$$

is invariant under the MI and MII moves. Further,

$$V_b^{\mathcal{B}}(t) = -(t^{1/2} + t^{-1/2}) V_L(t).$$

Proof. The invariance under Markov moves is shown above. We now prove its equality to the Jones polynomial as follows. First, we have:

$$V_{\mathrm{Id}}^{\mathcal{B}}(t) = \mathrm{Tr}(h) = -t^{1/2} - t^{-1/2}.$$

It is now sufficient to verify that (11) satisfies the same skein relation as the Jones polynomial. Indeed,

$$\begin{aligned} & t^{-1} R_1 - t R_1^{-1} \\ &= t^{-1} \begin{pmatrix} -t^{1/2} & 0 & 0 & 0 \\ 0 & 0 & -t & 0 \\ 0 & -t & t^{3/2} - t^{1/2} & 0 \\ 0 & 0 & 0 & -t^{1/2} \end{pmatrix} - t \begin{pmatrix} -t^{-1/2} & 0 & 0 & 0 \\ 0 & t^{-3/2} - t^{-1/2} & -t^{-1} & 0 \\ 0 & -t^{-1} & 0 & 0 \\ 0 & 0 & 0 & -t^{-1/2} \end{pmatrix} \\ &= (t^{1/2} - t^{-1/2}) \mathrm{Id}_{V \otimes V}. \end{aligned}$$

□

3.4 Burau representation and Alexander polynomial

The Burau representation of $\mathcal{B}_n$ is given by the R-matrix:

$$R_2 = \begin{pmatrix} t^{-1/2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & t^{-1/2} - t^{1/2} & 0 \\ 0 & 0 & 0 & -t^{1/2} \end{pmatrix}.$$

and the endomorphism:

$$h = \begin{pmatrix} t^{1/2} & 0 \\ 0 & -t^{-1/2} \end{pmatrix}$$

As for Jones polynomial, it is easy to verify that $\mathrm{Tr}_2((\mathrm{Id}_V \otimes h) \cdot R_2^{\pm 1}) = \mathrm{Id}_V$ and $(h \otimes h) \cdot R_2 = R_2 \cdot (h \otimes h)$. This implies that $\mathrm{Tr}(h^{\otimes n} \cdot \psi^{R_2}(\cdot))$ is invariant under MI and MII. However $\mathrm{Tr}(h^{\otimes n} \cdot \psi^{R_2}(\cdot))$ is always equal to 0. To obtain a non-trivial invariant from this representation, we consider a modification where we do not take the trace with respect to the first entry of $V^{\otimes n}$, as in the following theorem.

Theorem 3.4. Let L be an oriented link and $b \in \mathcal{B}_n$ a braid whose closure is isotopic to L. Then, for the above representation ψ_n, and the linear map h the following equation holds for some scalar c:

$$\mathrm{Tr}_{2,3,\ldots,n}\left((1 \otimes h^{\otimes(n-1)}\psi_n(b)\right) = c\ \mathrm{Id}_V,$$

where $\mathrm{Tr}_{2,3,\ldots,n}$ denotes the trace on the $2, 3, \ldots, n-$th entries of $V^{\otimes n}$. Further, c is an isotopy invariant of L. Moreover, c is equal to the Alexander polynomial of L.

References

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