Verdier quotient notes

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We use cohomological conventions:

$$C : \cdots \to C^{i-1} \to C^i \to C^{i+1} \to \cdots$$

1 Basic notions ([Wei94])

In this section following notions are detailed:

- cone (1.1)
- cylinder (1.3)
- quasi-isomorphisms (1.5)
- split short exact sequence (1.7)
- split (complex) (1.10)
- (co)chain homotopy (1.11)
- null-homotopic (1.12)
- (co)chain homotopy equivalence (1.14)

Definition 1.1. Let $f : B \to C$ be a map of cochain complexes. The *mapping cone* of f is the cochain complex $\mathrm{Cone}(f)$ whose degree n part is $B^{n+1} \oplus C^n$ and the differentials $d(b, c) = (-d(b), d(c) - f(b))$

Proposition 1.2. We have the following short exact sequence of complexes:

$$0 \to C \xrightarrow{v} \mathrm{Cone}(f) \xrightarrow{\delta} B[1] \to 0.$$

where $v : c \mapsto (0, c)$ and $\delta : (b, c) \mapsto -b$.

Definition 1.3. Let $f : B \to C$ be a map of cochain complexes. The *mapping cylinder* $cyl(f)$ is the cochain complex whose degree n part is $B_n \oplus B_{n+1} \oplus C_n$ and the differential is $d(b, b', c) = (d(b)+b', -d(b'), d(c)-f(b'))$.

Proposition 1.4. $cyl(f) = \mathrm{Cone}(g)$ where $g : \mathrm{Cone}(f)[-1] \to C$.

Definition 1.5. A morphism $B \to D$ of cochain of complexes is called a *quasi-isomorphism* if the maps $H_n(B) \to H_n(C)$ are all isomorphisms.

Proposition 1.6. If the cone of a map is acyclic ($H_n = 0$ for all n) then the map is a quasi-isomorphism.

Definition 1.7. A short exact sequence in $\mathcal{A}$ (an abelian category) $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ is called *split* if one of this condition hold :

(1) There exists a section of p, hence a morphism $s : C \to B$ such that $p \circ s = \mathrm{Id}_C$.
(2) There exists a retract of i, hence a morphism $r : B \to A$ such that $r \circ i = \mathrm{Id}_A$.
(3) There exists an isomorphim of sequences with the sequence :

$$0 \to A \xrightarrow{i} A \oplus C \xrightarrow{p} C \to 0$$

given by the direct sum and its canonical injection/projection morphisms.

Proposition 1.8. Let,

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of complex. This ses of complex split if and only if it exist $\delta : C[-1] \to A$ st $B \simeq \text{Cone}(\delta)$ and the short exact sequence is isomorphic to the usual ses:

$$0 \to A \to \text{Cone}(\delta) \to C \to 0$$

with the usual arrow. (remark that $C[-1][1] = C$)

Proof : You just need to define $\delta := d^A \circ r \circ s$, where r and s are the splitting maps that are comming from the splitting lemma. The other implication is trivial.

□

Lemma 1.9. (splitting lemma) The three condition of the last definition are equivalent.

Definition 1.10. A complex C is called *split* if there are maps $C_n \to C_{n-1}$ st $d = dsd$. The maps s_n are called the splitting maps. If in addition C is acyclic, we say that C is split exact.

$\mathcal{A}$ an abelian category.

Definition 1.11. We say that two cochain maps f and g from C to D are *cochain homotopic* if it exists maps $s_n : C^{n+1} \to D^n$ st:

$$f - g = sd + ds.$$

Definition 1.12. We say that a cochain map $f : C \to D$ is *null-homotopic* f is homotopic to 0 (isomorphic to 0 in $\boldsymbol{\mathcal{C}}(\mathcal{A})$).
C is said *null-homotopic* if the zero map $0 : C \to 0$ is an isomorphism in $\boldsymbol{\mathcal{C}}(\mathcal{A})$.

Remark 1.13. It's an equivalence relation (Ex 1.4.5 Weibel) noted $\sim_{htp}$

Definition 1.14. $f : C \to D$ is a *cochain homotopy equivalence* (isomorphism in $\boldsymbol{\mathcal{C}}(\mathcal{A})$) if it exist $g : D \to C$ st $fg \sim_{htp} \text{Id}_D$ and $gf \sim_{htp} \text{Id}_C$

Lemma 1.15. If f and g are cochain homotopic, then they induce the same map $H_n(C) \to H_n(D)$.

2 Derivated category ([Wei94])

$\mathcal{A}$ an abelian category.

2.1 The homotopy category $\boldsymbol{\mathcal{C}}(\mathcal{A})$

$\mathcal{A}$ an abelian category.

Definition 2.1. We have the following categories:

$\mathbf{Ch}(\mathcal{A})$:	$\boldsymbol{\mathcal{C}}(\mathcal{A})$:
Objects : complexes	Objects : complexes
Morphisms : complexes morphisms	Morphisms : $Hom_{\mathbf{Ch}(\mathcal{A})}(C, D)/\sim_{htp}$
Abelian category	Additive category

Proposition 2.2. Let $F : \mathbf{Ch}(\mathcal{A}) \to \mathcal{D}$ a functor that sends cochain homotopy equivalences to isomorphisms. Then F factors uniquely through $\boldsymbol{\mathcal{C}}(\mathcal{A})$:

$$\begin{array}{ccc} \mathbf{Ch}(\mathcal{A}) & \xrightarrow{F} & \mathcal{D} \\ \downarrow & \nearrow_{\exists !} & \\ \boldsymbol{\mathcal{C}}(\mathcal{A}) & & \end{array} \tag{1}$$

Definition 2.3. Let $f : C \to D$ a morphism in $\mathbf{Ch}(\mathcal{A})$. The *strict triangle* on f is the triple (f, v, δ) in $\boldsymbol{\mathcal{C}}(\mathcal{A})$ of the def of $\mathrm{Cone}(f)$.

$$\begin{array}{ccc} & \mathrm{Cone}(f) & \\ {\scriptstyle\delta}\swarrow & & \nwarrow{\scriptstyle v} \\ A & \xrightarrow{f} & B \end{array} \tag{2}$$

Now consider $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ in $\boldsymbol{\mathcal{C}}(\mathcal{A})$. We say that (u, v, w) is an *exact triangle* on (A, B, C) if it is "isomorphic" to a strict triangle in the sence that:

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & A[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\ A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & \mathrm{Cone}(u') & \xrightarrow{\delta} & A'[1] \end{array} \tag{3}$$

commuting in $\boldsymbol{\mathcal{C}}(\mathcal{A})$ and f, g and h are isomorphisms in $\boldsymbol{\mathcal{C}}(\mathcal{A})$.

It's usually writen as follows:

$$\begin{array}{ccc} & C & \\ {\scriptstyle w}\swarrow & & \nwarrow{\scriptstyle v} \\ A & \xrightarrow{u} & B \end{array} \tag{4}$$

Remark 2.4. Exact triangle of $\boldsymbol{\mathcal{C}}(\mathcal{A})$ are the short exact sequences that split (1.8).

2.2 Triangulated categories

Definition 2.5. Let $\mathcal{C}$ be a category equipped with an automorphism T (functor from $\mathcal{C}$ to $\mathcal{C}$). A *triangle* on an ordered triple (A, B, C) of objects of $\mathcal{C}$ is a triple (u, v, w) of morphisms, where $u : A \to B, v : B \to C$ and $w : C \to T(A)$
A *morphism of triangle* is a triple (f, g, h) forming a commutative diagram in $\mathcal{C}$:

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & T(A) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle T(f)} \\ A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & T(A') \end{array} \tag{5}$$

Definition 2.6. An additive category $\mathcal{C}$ is called a *triangulated category* if it is equipped with an automorphism $T : \mathcal{C} \to \mathcal{C}$ (translation functor) and a family of triangles (u, v, w) (called the *exact triangles* or *distinguished triangles* in $\mathcal{C}$), which are subject to the following four axioms:

(1) $\forall\, u : A \to B$ can be embedded in an exact triangle (u, v, w). If $A = B$ and $C = 0$, then the triangle $(\mathrm{Id}_A, 0, 0)$ is exact. If (u, v, w) is a triangle on (A, B, C), isomorphic to an exact triangle (u', v', w') on (A', B', C'), then (u, v, w) is also exact.

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & T(A) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & T(A') \end{array} \tag{6}$$

(2) (Rotation) If (u, v, w) as an exact triangle on (A, B, C), then both its "rotates" $(v, w, -Tu)$ and $(-T^{-1}(w), u, v)$ are exacts triangles for correspondant triplet.

(3) (Morphisms) Given two exact triangles

$$\begin{array}{ccc} & C & \\ {\scriptstyle w}\swarrow & & \nwarrow{\scriptstyle v} \\ A & \xrightarrow{u} & B \end{array} \qquad \begin{array}{ccc} & C' & \\ {\scriptstyle w'}\swarrow & & \nwarrow{\scriptstyle v'} \\ A' & \xrightarrow{u'} & B' \end{array} \tag{7}$$

with morphisms $f : A \to A'$, $g : B \to B'$ st $gu = u'f$, there exists a morphism $h : C \to C'$ so that (f, g, h) is a morphism of triangle.

(4) (The octahedral axiom) Given A, B, C, A', B', C' and (u, j, δ) on (A, B, C'), (v, x, i) on (B, C, A') and (vu, y, δ) on (A, C, B') three exact triangles. Then there is a fourth exact triangles:

$$\textit{comming soon} \tag{8}$$

Example 2.7.

- $\boldsymbol{\mathcal{C}}(\mathcal{A})$ is a triangulated category with exact triangles (of 2.3) and $T = [1]$.
- The *p*-homotopy category: $\mathcal{C}(A, \partial_0)$ (Category of p-chain complexes of A quotiented by the ideal of p-null homotopic morphims) is a triangulated category with $T(M) = M \otimes {}^{\Bbbk[\partial_0]}/_{\partial_0^{p-2}}\{p-1\}$ (see [QS22] p5)

Remark 2.8. If (u, v, w) is an exact triangle, then $v \circ u = w \circ v = Tu \circ w = 0$. (Ex 10.2.1 [Wei94]).

Remark 2.9. A short exact sequence $A \xrightarrow{u} B \xrightarrow{v} C$ in $\boldsymbol{\mathcal{C}}(\mathcal{A})$ gives not necessarily an exact triangle. Indeed, the map $w : C \to A[1]$ making (u, v, w) an exact triangle, may not be. The cohomological long exact sequence that arise from this short exact sequence can be obtained thanks the following exact triangle:

$$\begin{array}{ccc} & \mathrm{Cone}(u) & \\ \swarrow & & \nwarrow \\ A & \longrightarrow & cyl(u) \end{array} \tag{9}$$

and the quasi-isomorphisms $\beta : \left| \begin{array}{ccc} cyl(u) & \longrightarrow & B \\ (a, a', b) & \longmapsto & u(a) + b \end{array} \right.$ and $\phi : \left| \begin{array}{ccc} \mathrm{Cone}(u) & \longrightarrow & C \\ (a, b) & \longmapsto & v(b) \end{array} \right.$.

Definition 2.10. Let $\mathcal{C}$ and $\mathcal{D}$ be two triangulated cat. $F : \mathcal{C} \to \mathcal{D}$ is *triangulated functor* if it's an additive functor that commutes with the translation functor T and T', and sends exact triangles on exact triangles.

Definition 2.11. Let $\mathcal{D}$ a triangulated category. A *full additive subcategory* $\mathcal{C}$ in $\mathcal{D}$ is called a triangulated subcategory if every object isomorphic to an object of $\mathcal{C}$ is in $\mathcal{C}$ and the inclusion $\mathcal{C} \hookrightarrow \mathcal{D}$ is a triangulated functor. We assume further that

$$\phi_X : 1(T(X)) \to T(1(X))$$

is the identity on $T(X)$.

Definition 2.12. Let $\mathcal{C}$ be a triangulated category and $\mathcal{A}$ an abelian cat. An additive functor $H : \mathcal{C} \to \mathcal{A}$ is called a *cohomological functor* if whenever (u, v, w) is an exact triangle on (A, B, C) the long sequence:

$$\cdots \xrightarrow{w^*} H(T^i A) \xrightarrow{u^*} H(T^i B) \xrightarrow{v^*} H(T^i C) \xrightarrow{w^*} H(T^{i+1} A) \xrightarrow{u*} \cdots \tag{10}$$

is exact in $\mathcal{A}$.

Example 2.13.

- $H^0 : \boldsymbol{\mathcal{C}}(\mathcal{A}) \to \mathcal{A}$ where $\mathcal{A}$ is an abelian cat.
- $Hom_{\mathcal{C}}(\cdot, A) : \mathcal{C} \to \mathbf{Ab}$ where $\mathcal{C}$ is a triangulated cat.

2.3 The derived category

Definition 2.14. Let S be a collection of morphisms in a category $\mathcal{C}$. A *localization of $\mathcal{C}$ with respect to S* is a category $S^{-1}\mathcal{C}$, together with a functor $q : \mathcal{C} \to S^{-1}\mathcal{C}$ st:

(1) $q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$ for every s.
(2) Any functor $F : \mathcal{C} \to \mathcal{D}$ st $F(s)$ is an isomorphism for all $s \in S$ factor in a unique way through q. (It follow that $S^{-1}\mathcal{C}$ is unique up to equivalence.)

Example 2.15.

- A a ring. It can be seen as a category with one object and its morphisms are $End(A)$. Then localize A as a category is the same thing as localize A as a ring.
- $\boldsymbol{\mathcal{C}}(\mathcal{A}) = S^{-1}\mathbf{Ch}(\mathcal{A})$ where S is the collection of chain homotopy equivalence.

Definition 2.16. Let $\tilde{Q}$ be the collection of all quasi-isomorphisms in $\mathbf{Ch}(\mathcal{A})$ and Q the collection of quasi-isomorphisms in $\boldsymbol{\mathcal{C}}(\mathcal{A})$.

$$\mathbf{D}(\mathcal{A}) := \tilde{Q}^{-1}\mathbf{Ch}(\mathcal{A}) = Q^{-1}\boldsymbol{\mathcal{C}}(\mathcal{A}).$$

To prove the existence of the derived category in our context ($\mathcal{A}$ is a category of module) we need to introduce the notion of *multiplicative system* and *system that arises from a cohomological functor*. Thanks to this notions we can use the Gabriel-Zisman theorem (th 10.3.7 of [Wei94]). Moreover, in our context, we can prove that $\mathcal{D}(\mathcal{A})$ is a triangulated category (prop 10.4.1 [Wei94]).

3 Relative homotopy category

3.1 Verdier quotient ([Nee14])

Definition 3.1. Let $F : \mathcal{D} \to \mathcal{T}$ be a triangulated functor. The Kernel of F is defined to be the full subcategory $\mathcal{C}$ of $\mathcal{D}$ whose objects map to objects of $\mathcal{T}$ isomorphic to 0. In other words:

$$\mathcal{C} = \{x \in Obj(\mathcal{D}) \mid F(x) \text{ is isomorphic to } 0\}.$$

Proposition 3.2. The Kernel of F is a triangulated subcategory of $\mathcal{D}$.

Theorem 3.3. Let $\mathcal{D}$ be a triangulated category, $\mathcal{C} \subset \mathcal{D}$ a triangulated subcategory. Then there is a universal functor $F : \mathcal{D} \to \mathcal{T}$ with $\mathcal{C} \subset \mathrm{Ker}(F)$. In other words, there exists a triangulated category $\mathcal{D}/\mathcal{C}$ so that $\mathcal{C}$ is in the Kernel of F_{univ}, and F_{univ} is universal with this property. If $F : \mathcal{D} \to \mathcal{T}$ is a triangulated functor whose Kernel contains $\mathcal{C}$, then it factors uniquely as:

$$\mathcal{D} \xrightarrow{F_{univ}} \mathcal{D}/\mathcal{C} \to \mathcal{T}.$$

Remark 3.4. Description of $\mathcal{D}/\mathcal{C}$:
Objects: $\mathrm{Obj}(\mathcal{D})$ (F_{univ} is the identity on objects).
Morphism: For a triangulated subcategory $\mathcal{C} \subset \mathcal{D}$, we define a $Mor_{\mathcal{C}} \subset \mathcal{D}$. A morphism $f : X \to Y$ lies in $Mor_{\mathcal{C}}$ if and only if in the triangle:

$$X \xrightarrow{f} Y \to Z \to T(X)$$

the object Z lies in $\mathcal{C}$ (detail in section 1.5 of [Nee14]). Note that it is irrelevant which particular triangle we take because Z is unique up to isomorphism (rmk 1.1.21 [Nee14]). For two objects X, Y in $\mathcal{D}$ note $\alpha(X, Y)$ the class of diagrams of the form:

$$\begin{array}{ccc} & Z & \\ {}^{f}\swarrow & & \searrow^{g} \\ X & & Y \end{array} \tag{11}$$

st f belongs to $Mor_{\mathcal{C}}$. Consider a relation $R(X, Y)$ on $\alpha(X, Y)$ st $[(Z, f, g), (Z', f', g')]$ belongs to $R(X, Y)$ if and only if there is an element (Z'', f'', g'') in $\alpha(X, Y)$ and morphisms:

$$u : Z'' \to Z \;,\; v : Z'' \to Z'$$

which make the diagram

$$\begin{array}{ccccc} & & Z' & & \\ & {}^{f'}\swarrow & \uparrow{\scriptstyle v} & \searrow^{g'} & \\ X & \xleftarrow{f''} & Z'' & \xrightarrow{g''} & Y \\ & {}_{f}\nwarrow & \downarrow{\scriptstyle u} & \nearrow_{g} & \\ & & Z & & \end{array} \tag{12}$$

commute (With the previous definition, we can prove that u and v are in $Mor_{\mathcal{C}}$). Then $\mathrm{Hom}_{\mathcal{D}/\mathcal{C}}(X,Y)$ is the class of equivalence in $\alpha(X,Y)$ with respect to $R(X,Y)$. A morphism in $\mathcal{D}$, $f : X \to Y$ is sent to

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle 1}\downarrow & & \\ X & & \end{array} \tag{13}$$

Remark 3.5. Take $f : X \to Y$ st $f \in Mor_{\mathcal{C}}$. Then a triangulated functor F takes an object Z to 0 if and only if it takes the distinguished triangle

$$X \xrightarrow{f} Y \to Z \to T(X)$$

to a triangle isomorphic to the image of

$$X \xrightarrow{1} X \to 0 \to T(X)$$

In other words, $F(f)$ must be an isomorphism. Therefore, in the category $\mathcal{D}/\mathcal{C}$ all the morphisms in $Mor_{\mathcal{C}}$ will become invertible.

Proposition 3.6. The functor $F_{univ} : \mathcal{D} \to \mathcal{D}/\mathcal{C}$ is universal for all functor $F : \mathcal{D} \to \mathcal{T}$ which take all morphisms in $Mor_{\mathcal{C}}$ to invertible morphisms.

Remark 3.7. $\mathcal{D}/\mathcal{C} \cong (Mor_{\mathcal{C}})^{-1}\mathcal{D}$?

Remark 3.8. A distinguished triangle on $\mathcal{D}/\mathcal{C}$ is a triangle isomorphic to:

$$F_{univ}(X) \to F_{univ}(Y) \to F_{univ}(Z) \to TF_{univ}(X)$$

where $X \to Y \to Z \to TX$ is a distinguished triangle in $\mathcal{D}$.

3.2 Definition of the RHC

Let H be a Hopf algebra

Definition 3.9. *An algebra* in $\mathcal{C}$, a monoidal category, is a triple (A, m_A, η_A) where $A \in \mathrm{Obj}(\mathcal{C})$, $m_A : A \otimes A \to A$ (multiplication) and $\eta : 1_{\mathcal{C}} \to A$ (unit) st: m_A is associative and η_A is the unit for m_A. [BCPO19]

Definition 3.10. A is a *H-module algebra* if A is an algebra object in the module category of the Hopf algebra H. In particular (with the Sweedler's notation: $\Delta(h) = \sum_h h_1 \otimes h_2$):

$$h \cdot (ab) = \sum_h (h_1 \cdot a)(h_2 \cdot b) \quad \text{and} \quad h \cdot 1_A = \varepsilon(h) 1_A.$$

Definition 3.11. The *smash product algebra* $A\#H$ as an abelian group is isomorphic to $A \otimes H$ and the multiplicative structure is determined by:

$$(a \otimes h)(b \otimes k) = \sum_h a(h_1 \cdot b) \otimes h_2 k,$$

Remark 3.12. There is the forgetful functor between the usual homotopy categories of chain complexes of graded modules:

$$For : \boldsymbol{\mathcal{C}}(A\#H) \to \boldsymbol{\mathcal{C}}(A).$$

According to (3.1), an object $C.$ of $\boldsymbol{\mathcal{C}}(A\#H)$ is annihilated by the forgetful functor if and only if, when forgetting the H-module structure on each term of $C.$, the complex is null-homotpic (isomorphic to 0 in $\boldsymbol{\mathcal{C}}(A)$).

Definition 3.13. An object in $\mathrm{Ker}(For)$ (described above) is said *relatively null-homotopic.*

Definition 3.14. Given an H-module algebra A, the *relative homotopy category* is the Verdier quotient:

$$\mathcal{C}^H(A) := \frac{\mathcal{C}(A\#H)}{\mathrm{Ker}(For)}.$$

Remark 3.15. For reasons that are detailed in [QRSW23], if A is a field $\Bbbk$, then $\mathcal{C}^H(\Bbbk) = \mathcal{D}(H)$.

Thanks to the universal property of the Verdier quotient , we have the following diagram:

$$\begin{array}{ccc} \mathcal{C}(A\#H) & \xrightarrow{For} & \mathcal{C}(A) \\ & {\scriptstyle \mathcal{F}} \searrow \quad \nearrow & \\ & \mathcal{C}^H(A) & \end{array} \tag{14}$$

3.3 Triangulated structure on the RHC

Proposition 3.16. A triangle is distinguished in $\mathcal{C}^H(A)$ if and only if it is isomorphic to

$$\mathcal{F}(M) \to \mathcal{F}(N) \to \mathcal{F}(L) \to \mathcal{F}(M)[1]$$

where $M \to N \to L$ is a short exact sequence in $\mathcal{C}(A\#H)$ termwise A-split exact.

Proof : We begin by the first implication, a distinguished triangle in $\mathcal{C}^H(A)$ is a triangle isomorphic to

$$\mathcal{F}(M) \to \mathcal{F}(N) \to \mathcal{F}(L) \to \mathcal{F}(M)[1]$$

where $M \to N \to L \to M[1]$ is a distinguished triangle in $\mathcal{C}(A\#H)$. So $M \to N \to L$ is a short exact sequence of complexes that is termwise $A\#H$-split exact, especially it is termwise A-split exact. The first implication proved let us proved the converse.

Let $M \to N \to L$ be a short exact sequence of complexes in $\mathcal{C}(A\#H)$ that is termwise A-split exact. In the same category we have the distinguished triangle $M \to N \to \mathrm{Cone}(f) \to M[1]$ that gives rise to a distinguished triangle in $\mathcal{C}^H(A)$. So prove that $\mathrm{Cone}(f)$ and L are isomorphic in $\mathcal{C}^H(A)$ will show that our short exact sequence gives rise to a distinguished triangle in the relative homotopy category. So let us define

$$h : \left| \begin{array}{ccc} \mathrm{Cone}(f) & \longrightarrow & L \\ (m,n) & \longmapsto & g(n) \end{array} \right. ,$$

h is clearly surjective and $\mathrm{Ker}(h) = M[1] \oplus \mathrm{Ker}(g) = M[1] \oplus \mathrm{Im}(f)$. Thanks to the following diagram, we can assume that $Ker(h) \simeq \mathrm{Cone}(\mathrm{Id}_M)$ in $\mathcal{C}(A\#H)$:

$$\begin{array}{ccc} M_n \oplus Im(f)_{n-1} & \xrightarrow{\begin{pmatrix} d^M & 0 \\ f & d^N \end{pmatrix}} & M_{n+1} \oplus Im(f)_n \\ {\scriptstyle \begin{pmatrix} \mathrm{Id}_M & 0 \\ 0 & f \end{pmatrix}} \uparrow\downarrow & & \uparrow\downarrow {\scriptstyle \begin{pmatrix} \mathrm{Id}_M & 0 \\ 0 & f^{-1} \end{pmatrix}} \\ M_n \oplus M_{n-1} & \xrightarrow{\begin{pmatrix} d^M & 0 \\ \mathrm{Id} & d^N \end{pmatrix}} & M_{n+1} \oplus M_n \end{array} . \tag{15}$$

So we have the following short exact sequence $\mathrm{Cone}(\mathrm{Id}_M) \to \mathrm{Cone}(h) \xrightarrow{h} L$. Now under For we have:

$$For(\mathrm{Cone}(f)) \simeq For(M[1]) \oplus For(N) \simeq For(M[1]) \oplus For(M) \oplus For(L) \simeq For(\mathrm{Cone}(\mathrm{Id}_M)) \oplus For(L)$$

where the second isomorphism is gave by the fact that the original short exact sequence is termwise A-split. Thus we have the following distinguished triangle in $\mathcal{C}(A)$:

$$0 \simeq For(\mathrm{Cone}(\mathrm{Id}_M)) \to For(\mathrm{Cone}(f)) \xrightarrow{For(h)} For(L) \to For(\mathrm{Cone}(\mathrm{Id}_M)[1]) \simeq 0.$$

so $For(h)$ is an isomorphism.

Let $\text{Cone}(f) \xrightarrow{h} L \to Z \to \text{Cone}(f)[1]$ be a distinguished triangle in $\mathcal{C}A\#H)$, then $For(\text{Cone}(f) \xrightarrow{h} L \to Z \to \text{Cone}(f)[1])$ is a distinguished triangle in $\mathcal{C}(A)$ (because For is a triangulated functor) and because $For(h)$ is an isomorphism then $For(Z) \simeq 0$. In other words $Z \in \text{Ker}(For)$, thus $h \in Mor_{\text{Ker}(For)}$ and $\mathcal{F}(h)$ is an isomorphism in $\mathcal{C}^H(A)$. That conclude the proof.

$\square$

References

[BCPO19] D. Bulacu, Stefaan Caenepeel, Florin Panaite, and Freddy Oystaeyen. *Quasi-Hopf Algebras: A Categorical Approach*. Cambridge, 02 2019. doi:10.1017/9781108582780.

[Nee14] A. Neeman. *Triangulated Categories*. Annals of Mathematics Studies. Princeton University Press, 2014. URL: https://books.google.fr/books?id=VvkcBAAAQBAJ.

[QRSW23] You Qi, Louis-Hadrien Robert, Joshua Sussan, and Emmanuel Wagner. Symmetries of equivariant khovanov-rozansky homology, 2023. URL: https://arxiv.org/abs/2306.10729, arXiv:2306.10729.

[QS22] You Qi and Joshua Sussan. On some p-differential graded link homologies. *Forum of Mathematics, Pi*, 10:e26, 2022. arXiv:2009.06498 [math]. URL: http://arxiv.org/abs/2009.06498, doi:10.1017/fmp.2022.19.

[Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994.