Master's thesis of mathemitics

at IMJ-PRG (Paris)

Symmetries of equivarient Khovanov-Rozansky Homology

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1 Introduction

In order to access the categorification of the WRT invariants, one of the current works is to endow the categorification of link invariants with an additional algebraic structure. In [QRSW23] Qi, Robert, Sussan and Wagner endowed Khovanov-Rozansky homology with an action of the algebra $\mathfrak{sl}_2$ but in [QRSW24], it is implied that this action would come from the action of a larger Lie algebra, the positive half algebra of Witt $\mathfrak{W}_{-1}^{\infty}$.

In this work we present the $\mathfrak{sl}_2$ structure put on the Khovanov-Rozansky homology (section 5) by defining an $\mathfrak{sl}_2$-action on foams and state space (section 3.3) then correcting it thanks to green dots (section 4.2). At the same time we will work on the action of $\mathfrak{W}_{-1}^{\infty}$ on foams and state-space which is a first work in order to equip the Khovanov-Rozansky homology of a $\mathfrak{W}_{-1}^{\infty}$-action.

The logical continuation of this master's thesis is to define the Khovanov-Rozansky $\mathfrak{gl}_N$-homology equipped with the action of the Holf algebra $\mathcal{U}(\mathfrak{W}_{-1}^{\infty})$. For this, some relations on Witt green dots need to be found like green-dots migrations (lemma 4.3) or floating green dots.

Conventions

Let $\Bbbk$ be a ring with unity. For all $x \in \Bbbk$ we set $\overline{x} := 1 - x$.

Throughout most of the paper we will fix a natural number N.

The algebras $\mathbb{Z}_N := \mathbb{Z}[X_1, \dots, X_N]^{\mathfrak{S}_N}$ and $\Bbbk[X_1, \dots, X_N]^{\mathfrak{S}_N}$ of symmetric polynomials will play a central role in this paper. They are non-negatively graded by imposing that $\deg(X_i) = 2$. The ith elementary, complete homogeneous, and power sum symmetric polynomials in $X_1, \dots, X_N$ are denoted by E_i, H_i and P_i respectively.

Throughout most of this paper we invert 2 in the ground ring.

For $k \in \mathbb{N}$ we denote by Sym_k the set of symmetric polynomials with $\mathbb{Z}$ coefficients in k variables. In particular $\mathbb{Z}_N = Sym_N$. When working in such a ring, we will let e_i, h_i and p_i be the ith elementary, complete homogeneous, and power sum symmetric polynomials respectively without reference to the variables. The ring Sym_k is graded by imposing that e_i is degree $2i$. We denote by $\underline{X}_A = (X_j)_{j \in A}$ where A is a subset of $\{1, \dots, N\}$.

For a $\mathbb{Z}$-graded vector space V, let V_i denote the subspace in degree i. Let $q^n V$ denote the $\mathbb{Z}$-graded vector space where $(q^n V)_i = V_{i-n}$.

For $n \in \mathbb{Z}$, let $[n] := \frac{q^n - q^{-n}}{q - q^{-1}} = \sum_{i=0}^{n-1} q^{n-1-2i}$, for $k \in \mathbb{N}$ we let $[k]! = \prod_{j=1}^{k} [j]$. Finally, for $m \in \mathbb{Z}$ and $a \in \mathbb{N}$, define:

$$\begin{bmatrix} m \\ a \end{bmatrix} = \prod_{i=1}^{a} \frac{[m+1-i]}{[i]}.$$

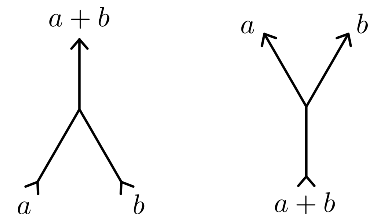
Note that if m is non-negative, one has $\begin{bmatrix} m \\ a \end{bmatrix} = \frac{[m]!}{[a]![m-a]!}$.

Complexes will be taken to be cohomologically graded, in other words the ith differential goes from C_i to C_{i+1}. For a complex C, we let $t^i C$ denote the shifted complex whose piece in cohomological degree $i + j$ is the piece of C in cohomological degree j, in other words: $(t^j C)_i = C_{i-j}$.

2 Topological preliminaries

2.1 Webs and Foams

Definition 2.1 Let Σ be a surface. A *closed web* (also called *web*) is a finite, oriented, trivalent graph $\Gamma = (V(\Gamma), E(\Gamma))$ (where $V(\Gamma)$ is the set of vertices and $E(\Gamma)$ the set of edges) embedded in the interior of Σ and endowed with a *thickness function* $\ell : E(\Gamma) \to \mathbb{N}$ satisfying a flow condition: vertices and thicknesses of their adjacent edges must be one of these two types:



(1)

The first type of vertex is called a *merge* vertex, the second a *split* vertex. In each of these types, there is one *thick* edge and two *thin* edges. Oriented circles with non-negative thickness are regarded as edges without vertices and can be part of a web. The embedding of Γ in Σ is smooth outside its vertices, and at the vertices should fit with the local models in (1)

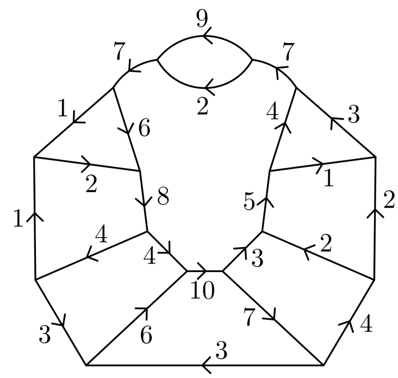


Figure 1: Example of web in $\mathbb{R}^2$

Definition 2.2 Let M be an oriented smooth 3-manifold with a collared boundary. A *foam* $F \subset M$ is a collection of *facets*, that are compact oriented surfaces labeled with non-negative integers and glued together along their boundary points, such that every point p of F has a closed neighborhood homeomorphic to one of the following (shows in Figure 2):

(1) a disk, when p belongs to a unique facet,

(2) $Y \times [0, 1]$, where Y is the neighborhood of a merge or split vertex of a web, when p belongs to three facets,

(3) the cone over the 1-skeleton of a tetrahedron with p as the vertex of the cone, so that it belongs to six facets.

The set of points of second type is a collection of curves called *bindings* and points of the third type are called *singular vertices*. The boundary ∂F of F is the closure of the set of boundary points of facets that do not belong to a biding. It is understood that F coincides with $\partial F \times [0, 1]$ on the collar of ∂M. For each facet f of F, we denote by $\ell(f) \in \{1, \ldots, N\}$ the *thickness* of f. A foam F is *decorated* if each facet f of F is assigned a symmetric polynomial $P_f \in Sym_{\ell(f)}$. In the second local model, it is implicitly understood that thicknesses of the three facets are given by that of the edges in Y. In particular, it satisfies a flow condition and locally one has a thick facet and two thin ones. We also require that orientations of bindings are induced by that of the thin facets and by the opposite of the thick facet. Foams are regarded up to ambient isotopy relative to boundary. Foams without boundary are said to be *closed*.

Remark 2.3 Decorations on facets are depicted by dots adorned with symmetric polynomials in

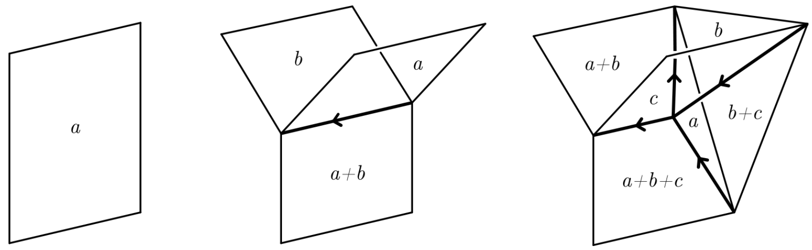


Figure 2: The three local models of a foam. The model in the middle we will denoted $Y^{(a,b)}$ and the model on the right is denoted $T^{(a,b,c)}$.

the correct number of variables (the sickness of the facet they sit on). The decoration is the product of all adornments of dots sitting on the facet. An other notation will be established later.

Notations 2.4 For a foam F, we write:

- F^2 for the collection of facets of F,
- F^1 for the collection of bindings,
- F^0 for the collection of singular vertices of F.

We partition F^1 as follows: $F^1 = F^1_o \sqcup F^1_-$, where F^1_o is the collection of circular bindings and F^1_- is the collection of bindings diffeomorphic to intervals.

Now we will define the N-degree of a foam: If $s \in F^1_-$ any of its points has a neighborhood diffemorphic to $Y^{(a,b)}$ for a given a and b and we define:

$$\deg_N(s) = ab + (a+b)(N-a-b).$$

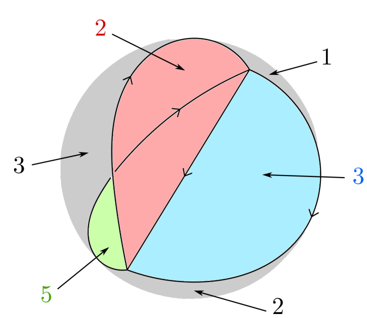
If $v \in F_0$, it has a neighborhood diffeomorphic to $T^{(a,b,c)}$ for a given a, b and c and we write:

$$\deg_N(v) = ab + bc + ac + (a+b+c)(N-a-b-c).$$

Definition 2.5 Let F be a decorated foam and suppose that all decorations are homogeneous. For all N in $\mathbb{N}$, the N-*degree* of F is the integer $\deg_N(F) \in \mathbb{Z}$ given by the following formula:

$$\begin{aligned}\deg_N(F) := &\sum_{f\in F^2} (\deg(P_f) - \ell(f)(N-\ell(f))\chi(f)) \\ &+ \sum_{s\in F^1_-} \deg_N(s) + \sum_{v\in F^0} \deg_N(v).\end{aligned}$$

Example 2.6



(2)

In this example the foam has a $N-$degree equal to -26.

In the case $M = \Sigma \times [0,1]$, for all foam F define in M, a generic section of it: $F_t = F \cap \{t\}$ with $t \in [0,1]$ is a web. We call F_0 and F_1 the *input* and the *output* of F respectively.
Now, for a given surface Σ, we define the category $Foam_\Sigma$ as follow:

- Objects are webs in Σ.
- $\mathrm{Hom}_{Foam_\Sigma}(\Gamma_0, \Gamma_1) = \{$decorated foams F in $\Sigma \times [0,1]$ where $F_0 = -\Gamma_0$ and $F_1 = \Gamma_1\}$

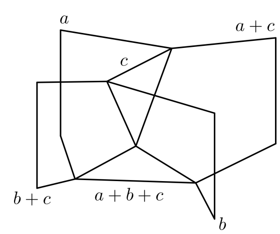
where $-\Gamma_0$ is the same web as Γ_0 but every edges have the opposite orientation. Composition is given by stacking foams on one another and rescaling. Decorations behave multiplicativly. The identity of Γ for this composition is the foam $\Gamma \times [0,1]$ decorated with the constant polynomial 1 on every facet. The N-degree is additive under composition.

If Γ is a web in a surface Σ, and $h : \Sigma \times [0,1] \to \Sigma$ is a smooth isotopy of Σ, one can define $F(h)$, the trace of $h(\Gamma)$ in $\Sigma \times [0,1]$, by: for all $t \in [0,1]$ $F(h)_t = h_t(\Gamma)$. Such foams are called *traces of isotopies*. They have degree 0.

Definition 2.7 A foam in a surface $\Sigma \times [0,1]$ is *basic* if it is a trace of isotopy or if it is equal to $\Gamma \times [0,1]$ outside a cylinder $B \times [0,1]$, and where it is given inside by one of the local models given in Figure 3. A foam in $\Sigma \times [0,1]$ is in *good position* if it is a composition of basic foams. If Γ is a web in $\mathbb{R}^2$ we denote by $V(\Gamma)$ the free $\Bbbk$-module generated by foams in good position in $\mathbb{R}^2 \times [0,1]$ with $\varnothing$ as input and Γ as output.

Remark 2.8 Every foam in $\Sigma \times [0,1]$ is isotopic to a foam in good position.

Example 2.9 The following foam in good position will come up later in section 4.3. It is the composition of digon cup associativity and unzip foams with adequat thicknesses.



2.2 $\mathfrak{gl}_N$ foam evaluation

Definition 2.10 A *pigment* is an element of $\mathbb{P} = \{1, \dots, N\}$. If A is a subset of $\mathbb{P}$, $\overline{A}$ denotes $\mathbb{P} \setminus A$. The set $\mathbb{P}$ is endowed with the canonical order.
A *coloring* of a foam F is a map $c : F^2 \to \mathcal{P}(\mathbb{P})$, such that:

- For each facet $f \in F^2$, the number of pigment on f, $\#c(f)$, is equal to $\ell(f)$.
- For each binding joining a facet f_1 with thickness a, a facet f_2 with thickness b, and a facet f_3 with thickness $a + b$, we have $c(f_1) \sqcup c(f_2) = c(f_3)$. This condition is called *flow condition*.

A foam which endowed a coloring is called a *colored foam*. A facet of thickness 0 is colored by the empty set.

Lemma 2.11 If (F, c) is a colored foam, i and j are two distinct element of $\mathbb{P}$ then,

(1) The union of all facets (with the identification coming from the gluing) of all the facets which contain the pigment i in their colors is a surface. It is called the *monochrome surface* of (F, c) associated with i and it is denoted $F_i(c)$. The restriction we imposed on the orientations of facets ensures that $F_i(c)$ is oriented.

(2) The union of all facets which contain i or j but not both in their colors is a surface. It is called the bichrome surface of (F, c) associated with i, j. It is the symmetric difference of

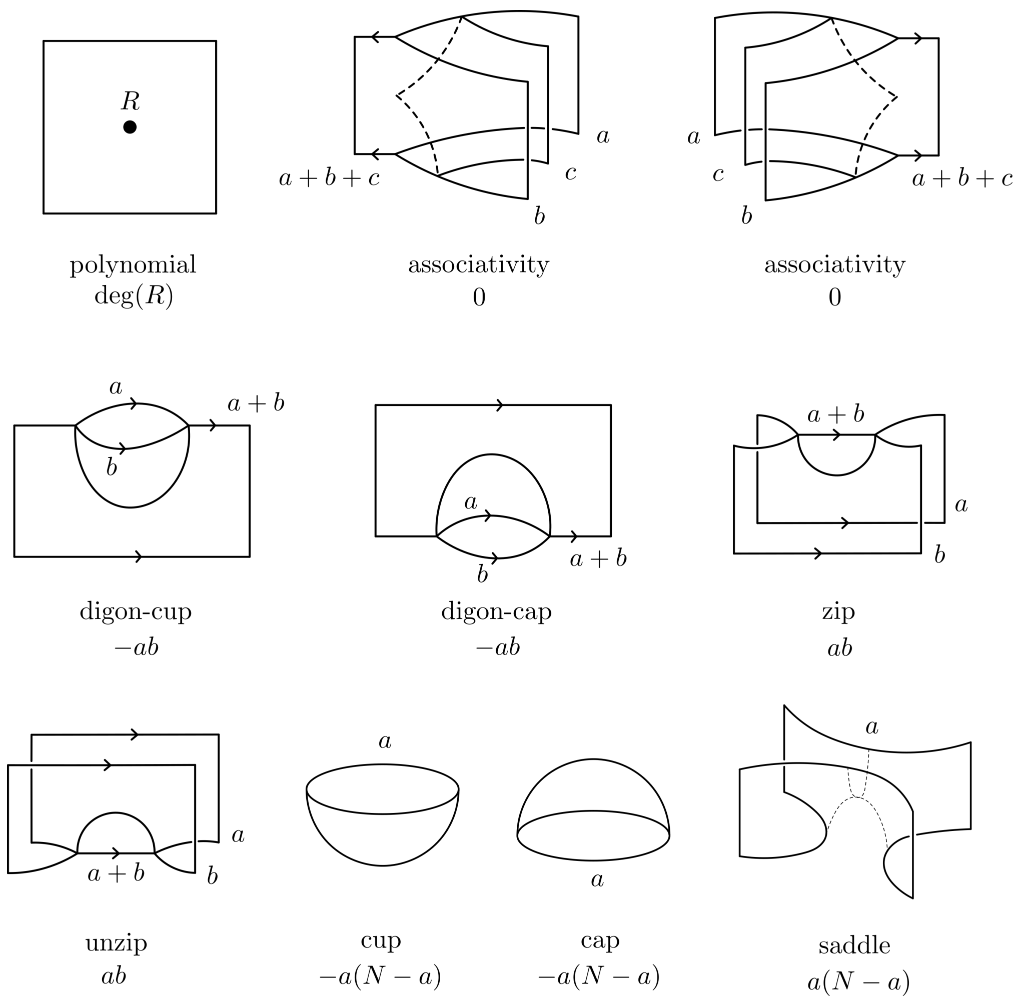


Figure 3: Local basic foams models with their degrees below.

$F_i(c)$ and $F_j(c)$ and it is denoted $F_{ij}(c)$. The restriction we imposed on the orientations on facets ensures that $F_{ij}(c)$ can be oriented by taking the orientations of the facets containing i and the reverse orientations of the facets containing j.

(3) We may assume $i < j$. We consider a binding joining the facets f_1, f_2 and f_3. Suppose that i is in $c(f_1)$, j is in $c(f_2)$ and i and j are in $c(f_3)$. We say that the binding is positive with respect to (i, j) if the cyclic order on the binding is (f_1, f_2, f_3) and negative with respect to (i, j) otherwise. The set $F_i(c) \cap F_j(c) \cap F_{ij}(c)$ is a collection of disjoints circles. Each of these circles is a union of bindings, for every circle the bindings are either all positive or all negative with respect to (i, j). We denote by $\theta^+_{i,j}$ the number of positive circle or $\theta^-_{i,j}$ the number of negative circle.

Given F, a decorated closed foam, and c a coloring of F, the *colored* $\mathfrak{gl}_N$*-evaluation* of (F, c) is the rational function in variables $X_1, \dots, X_N$ defined by:

$$\langle F, c\rangle_N := (-1)^{s(F,c)} \frac{P(F, c)}{Q(F, c)} \tag{3}$$

with

$$P(F,c) := \prod_{f \in F^2} P_f(\underline{X}_{c(f)}) \tag{4}$$

and

$$Q(F,c) := \prod_{1 \le i < j \le N} (X_i - X_j)^{\chi(F_{ij}(c))/2} \tag{5}$$

and where we have the following.

- $P_f(\underline{X}_{c(f)})$ is the evaluation of the polynomial P_f in the indeterminates $\underline{X}_{c(f)}$.
- $s(F,c)$ is the integer given by the following formula:

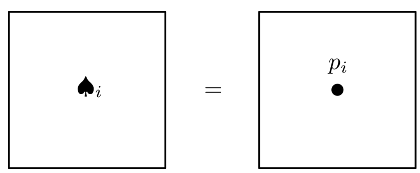
$$s(F,c) = \sum_{i=1}^{N} \frac{i\chi(F_i(c))}{2} + \sum_{1 \le i < j \le N} \theta_{ij}^{+}(F,c).$$

Finally we define the $\mathfrak{gl}_N$-*evaluation* of a foam F by:

$$\langle F \rangle_N := \sum_c \langle F, c \rangle_N. \tag{6}$$

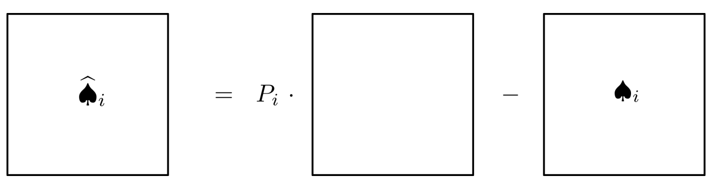
where the sum runs over all colorings of F.

In the following sections, decorations of foams that we consider will often be power sums, and so we introduce the following notation:



(7)

In particular, on a facet of thickness a, $\spadesuit_0 = a$. We will also extend the decorations allowed thanks to the decoration $\widehat{\spadesuit}_i$ which represent the ith power sum in the variable which are not in the facet. In other words:



(8)

Proposition 2.12 ([RW18, Prop 2.18]) Let F be a foam, then $\langle F \rangle(X)$ is a symmetric polynomial and if decorations are homogenous,

$$\deg(\langle F, c \rangle_N) = \deg(\langle F \rangle_N) = \deg_N(F). \tag{9}$$

Corollary 2.13 Let F be a foam with trivial decoration and c a coloring of F. The following identity holds:

$$\deg_N(F) = -\sum_{1 \le i < j \le N} \chi(F_{ij}(c)). \tag{10}$$

3 Action on foams

In the following section, we define two actions of Lie algebras on foams. First we briefly recall how these Lie algebra are defined and some interesting properties.

3.1 Two Lie algebras

Definition 3.1 Let $\mathfrak{sl}_2$ be the Lie algebra over $\Bbbk$ generated by $\mathtt{e}, \mathtt{f}$ and $\mathtt{h}$ with relations:

$$[\mathtt{h}, \mathtt{e}] = 2\mathtt{e}, \quad [\mathtt{h}, \mathtt{f}] = -2\mathtt{f}, \quad [\mathtt{e}, \mathtt{f}] = \mathtt{h}. \tag{11}$$

Definition 3.2 Let $\mathfrak{W}$ be the Lie algebra generated by $(L_n)_{n\in\mathbb{Z}}$ with relations $\forall n, m \in \mathbb{Z}$:

$$[L_n, L_m] = (n-m)L_{n+m}. \tag{12}$$

This algebra is called *Witt algebra*. Let us denote $\mathfrak{W}_{-1}^{\infty}$, the Lie subalgebra generated by symbols $(L_n)_{n\geqslant -1}$.

Proposition 3.3 The map:

$$\iota : \begin{cases} \mathtt{e} & \mapsto L_{-1} \\ \mathtt{h} & \mapsto 2L_0 \\ \mathtt{f} & \mapsto -L_1 \end{cases} \tag{13}$$

induces a morphism of Lie algebras from $\mathfrak{sl}_2$ to $\mathfrak{W}$ whose image is in $\mathfrak{W}_{-1}^{\infty}$. If 2 is not a zero divisor in $\Bbbk$, the map is injective.

Proof : We just need to verify that relations given in (11) are satisfied by $\iota(\mathtt{e}), \iota(\mathtt{h})$ and $\iota(\mathtt{f})$:

$$\begin{aligned} [\iota(\mathtt{h}), \iota(\mathtt{e})] &= [2L_0, L_{-1}] = 2L_{-1} = \iota(2\mathtt{e}) \\ [\iota(\mathtt{h}), \iota(\mathtt{f})] &= [2L_0, -L_1] = 2L_1 = \iota(-2\mathtt{f}) \\ [\iota(\mathtt{e}), \iota(\mathtt{f})] &= [L_{-1}, -L_1] = 2L_0 = \iota(\mathtt{h}) \end{aligned}$$

□

$\mathfrak{W}_{-1}^{\infty}$ acts on polynomial rings as follows. For any $Q \in \Bbbk[x_1, \dots, x_k]$:

$$L_n \cdot Q := -\sum_{i=0}^{k} x_i^{n+1} \frac{\partial Q}{\partial x_i} \tag{14}$$

Notes that we can restrict this action on $\Bbbk[x_1, \dots, x_k]^{\mathfrak{S}_k}$. Therefore, thanks to the proposition 3.3 $\mathfrak{sl}_2$ acts also on $\Bbbk[x_1, \dots, x_k]^{\mathfrak{S}_k}$ via these relations:

$$\mathtt{e} \cdot Q = -\sum_{i=0}^{k} \frac{\partial Q}{\partial x_i} \tag{15}$$

$$\mathtt{h} \cdot Q = -\deg(Q)\ Q \tag{16}$$

$$\mathtt{f} \cdot Q = \sum_{i=0}^{k} x_i^2 \frac{\partial Q}{\partial x_i} \tag{17}$$

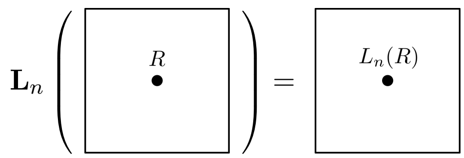
Definition 3.4 A *Witt-sequence* $(\lambda_n)_{n\geq -1}$ is a sequence such that $\lambda_{-1} = 0$ and for any $m, n \in \mathbb{N}$,

$$n\lambda_n - m\lambda_m = (n-m)\lambda_{m+n}.$$

For example, for any $\lambda \in \Bbbk$, the sequence given by $\lambda_n = \lambda(n+1)$ is a Witt sequence.

3.2 Action of $\mathfrak{W}_{-1}^{\infty}$ on foams

For the rest of this section, fix an element $s \in \Bbbk$, and two Witt-sequences $(\lambda_n)_{n\in\mathbb{N}_{-1}}$ and $(\nu_n)_{n\in\mathbb{N}_{-1}}$. We now define a sequence of operators $(\mathbf{L}_n)_{n\in\mathbb{N}_{-1}}$ acting on basic foams. For $n \in \mathbb{N}_{-1}$:



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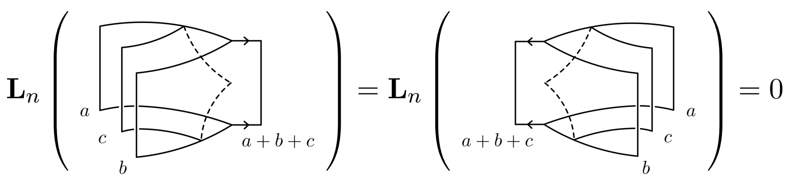

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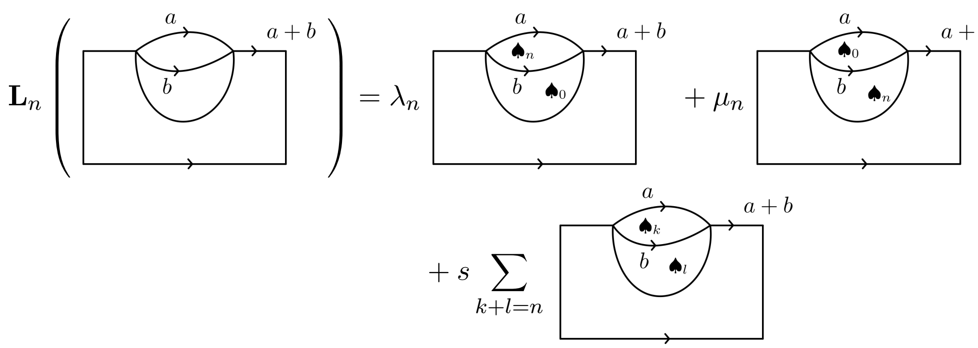

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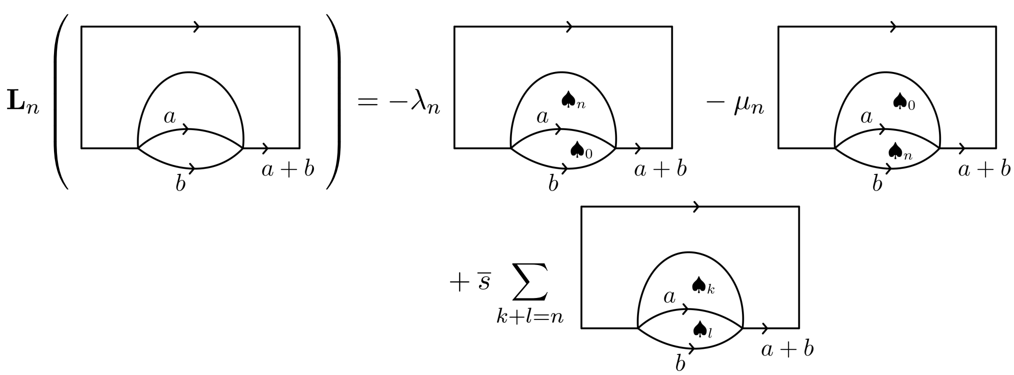

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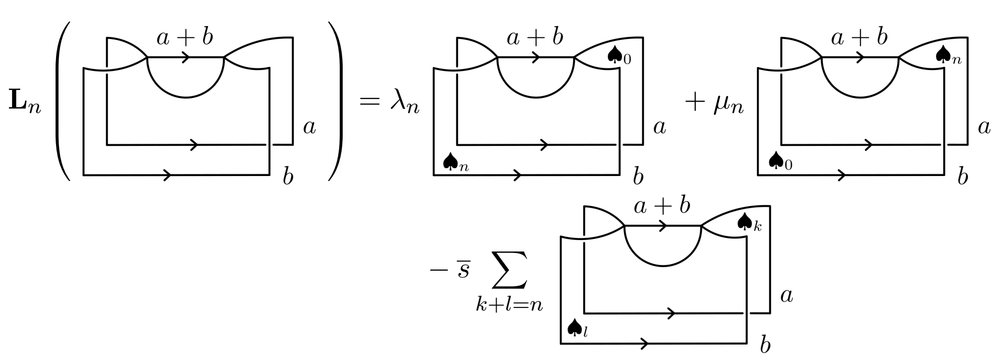

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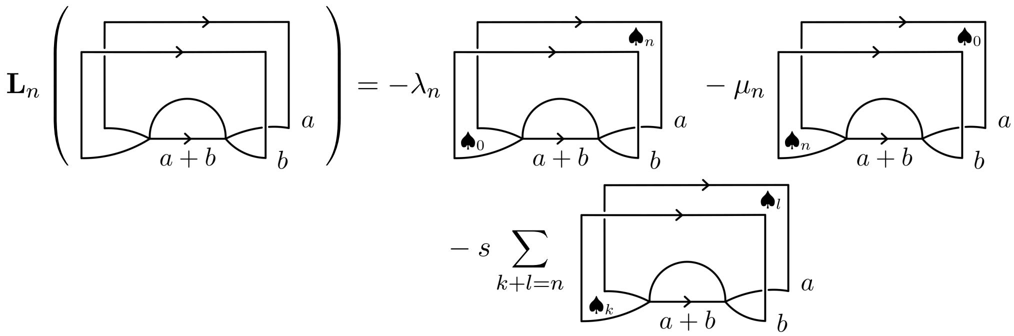

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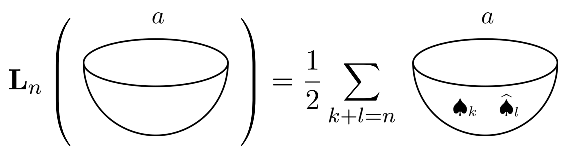

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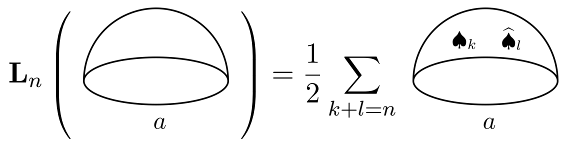

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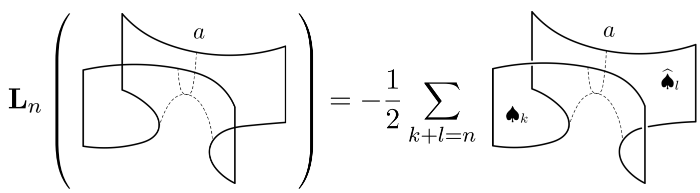

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Lemma 3.5 Mapping L_n to $\mathbf{L}_n$ for all $n \in \mathbb{N}_{-1}$ defines an action of $\mathfrak{W}_{-1}^{\infty}$ on the $\Bbbk$-module generated by foams in good position.

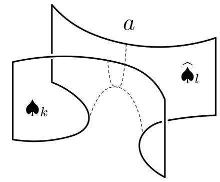
Proof. We need to prove that for all $n, m \in \mathbb{N}_{-1}$, $[\mathbf{L}_n, \mathbf{L}_m] = (n-m)\mathbf{L}_{n+m}$ where $[\mathbf{L}_n, \mathbf{L}_m] := \mathbf{L}_n \circ \mathbf{L}_m - \mathbf{L}_m \circ \mathbf{L}_n$. Without loss of generality, we can assume that $n \leq m$. For any foams F and G in good positions we have:

$$\begin{aligned}
[\mathbf{L}_n, \mathbf{L}_m](F \circ G) &= \mathbf{L}_n(\mathbf{L}_m(F) \circ G + F \circ \mathbf{L}_m(G)) - \mathbf{L}_m \circ (\mathbf{L}_n(F) \circ G + F \circ \mathbf{L}_n(G)) \\
&= ((\mathbf{L}_n \circ \mathbf{L}_m(F) - \mathbf{L}_m \circ \mathbf{L}_n(F)) \circ G + F \circ (\mathbf{L}_n \circ \mathbf{L}_m(G) - \mathbf{L}_m \circ \mathbf{L}_n(G)) \\
&= [\mathbf{L}_n, \mathbf{L}_m](F) \circ G + F \circ [\mathbf{L}_n, \mathbf{L}_m](G).
\end{aligned}$$

So the operator $[\mathbf{L}_n, \mathbf{L}_m]$ satisfies the Leibniz rules and $(n-m)\mathbf{L}_{n+m}$ as well. Therefore it is enough to prove the relation for basic foams. For traces of isotopies, this is trivial. For polynomial foams, the identity is satisfy because $\mathfrak{W}_{-1}^{\infty}$ act on polynomial foams as it acts on symmetric polynomials. So, we need to check that the identity hold on caps, cups, zips, unzips, saddles, digon-caps and digon-cups foams. In all case this is just a computation, we treat the case of saddle:
We noticed that for all $n, k \in \mathbb{N}$,

$$L_n(\spadesuit_k) = -k\spadesuit_{k+n} \quad \text{and} \quad L_n(\widehat{\spadesuit}_k) = -k\widehat{\spadesuit}_{k+n}.$$

We set:

$S_{k,l} :=$ 

and S is the saddle without decorations. Therefore:

$$\mathbf{L}_n \circ \mathbf{L}_m(S) = -\frac{1}{2} \sum_{k+l=m} \mathbf{L}_n(S_{k,l})$$

and,

$$\mathbf{L}_n(S_{k,l}) = -kS_{k+n,l} - lS_{k,l+n} - \frac{1}{2} \sum_{i+j=n} S_{(i,k),(l,j)}.$$

$S_{(i,k),(l,j)}$ denote the saddle decorated by $\spadesuit_i, \spadesuit_k, \widehat{\spadesuit}_l, \widehat{\spadesuit}_j$. Moreover, $\sum_{k+l=m} \sum_{i+j=n} S_{(i,k),(l,j)}$ is symmetric in n and m so:

$$\mathbf{L}_n \circ \mathbf{L}_m(S) - \mathbf{L}_m \circ \mathbf{L}_n(S) = \frac{1}{2} \sum_{k+l=m} kS_{k+n,l} + lS_{k,l+n} - \frac{1}{2} \sum_{k+l=n} kS_{k+m,l} + lS_{k,l+m}.$$

Now we distinguish three cases to established the coefficient of $S_{i,j}$ in the previous formula:

- if $0 \leq i \leq n$ then $m \leq j \leq n+m$ and the coefficient of $S_{i,j}$ is: $\frac{j-n}{2} - \frac{j-m}{2} = -\frac{n-m}{2}$
- if $n \leq i \leq m$ then $n \leq j \leq m$ and the coefficient of $S_{i,j}$ is: $\frac{i-n}{2} + \frac{j-n}{2} = \frac{i+j-2n}{2} = -\frac{n-m}{2}$
- if $m \leq i \leq n+m$ then $0 \leq j \leq n$ and the coefficient of $S_{i,j}$ is: $\frac{i-n}{2} - \frac{i-m}{2} = \frac{n-m}{2}$.

Therefore,

$$[\mathbf{L}_n, \mathbf{L}_m](S) = -\frac{n-m}{2} \sum_{i+j=n+m} S_{i,j} = (n-m)\mathbf{L}_{n+m}(S)$$

□

The computation for a cup is detailed on [QRSW24, p24], but the action is defined on spherical foams, in our framework you need to set $\nu_n = 0$.

Proposition 3.6 Let F be a decorated closed foam, then for all $n \in \mathbb{N}_{-1}$

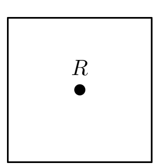
$$\langle L_n \cdot F \rangle_N = L_n \cdot \langle F \rangle_N \tag{31}$$

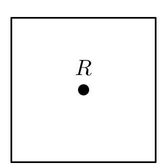
3.3 Action of $\mathfrak{sl}_2$ on foams

Thanks to Proposition 3.3 we can convert all results of the previous section on $\mathfrak{W}_{-1}^{\infty}$ and convert them on $\mathfrak{sl}_2$.

We fix $t_1 = \lambda_1 + s$ and $t_2 = \mu_1 + s$ by using the last section, we can have define $\mathbf{e} := \mathbf{L}_{-1}, \mathbf{h} := 2\mathbf{L}_0$ and $\mathbf{f} := \mathbf{L}_1$. These three operators verify the following relations:

The operator $\mathbf{e}$ acts via $-\sum_i \frac{\partial}{\partial x_i}$ on polynomials and by 0 on any other basic foam. The operators $\mathbf{h}$ and $\mathbf{f}$ are defined as follows:

$$\mathbf{h}\left(\vphantom{\int}\right.$$ 

$$\left.\vphantom{\int}\right) = -\deg(R)$$ 

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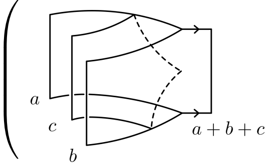
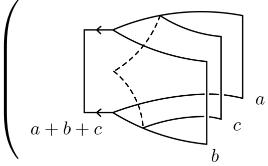
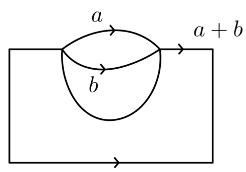
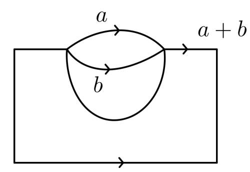
$$\mathbf{h}\Bigg($$ 

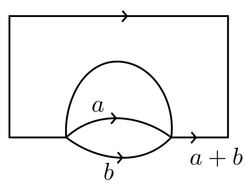
$$\Bigg) = \mathbf{h}\Bigg($$ 

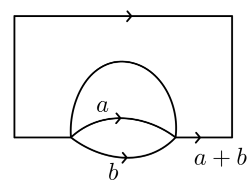
$$\Bigg) = 0 \tag{33}$$

$$\mathbf{h}\Bigg($$ 

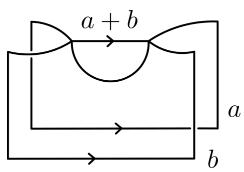
$$\Bigg) = ab(t_1 + t_2)$$ 

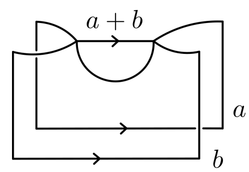
$$\tag{34}$$

$$\mathbf{h}\Bigg($$ 

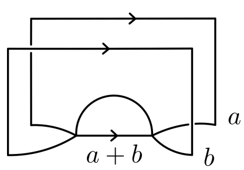
$$\Bigg) = ab(\overline{t_1} + \overline{t_2})$$ 

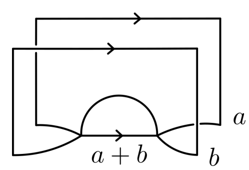
$$\tag{35}$$

$$\mathbf{h}\Bigg($$ 

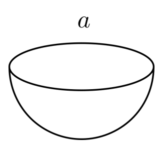
$$\Bigg) = -ab(\overline{t_1} + \overline{t_2})$$ 

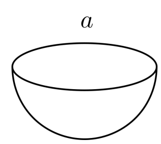
$$\tag{36}$$

$$\mathbf{h}\Bigg($$ 

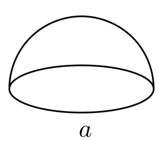
$$\Bigg) = -ab(t_1 + t_2)$$ 

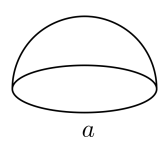
$$\tag{37}$$

$$\mathbf{h}\Bigg($$ 

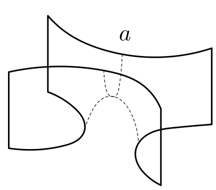
$$\Bigg) = a(N - a)$$ 

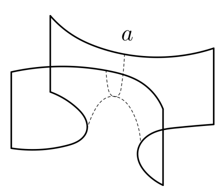
$$\tag{38}$$

$$\mathbf{h}\Bigg($$ 

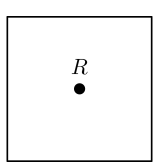
$$\Bigg) = a(N - a)$$ 

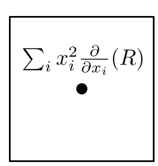
$$\tag{39}$$

$$\mathbf{h}\Bigg($$ 

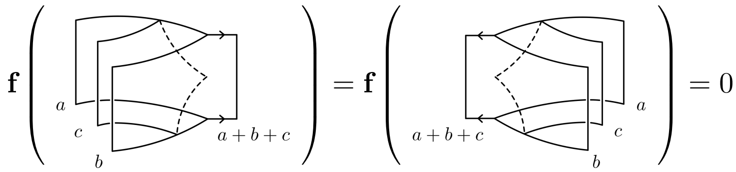
$$\Bigg) = -a(N - a)$$ 

$$\tag{40}$$

$$\mathbf{f}\Bigg($$ 

$$\Bigg) =$$ 

$$\tag{41}$$



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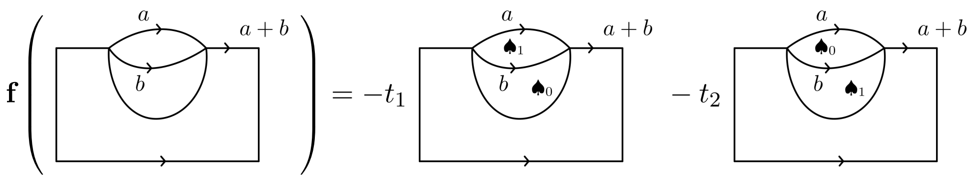

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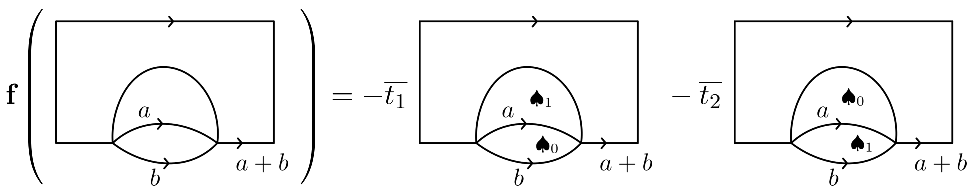

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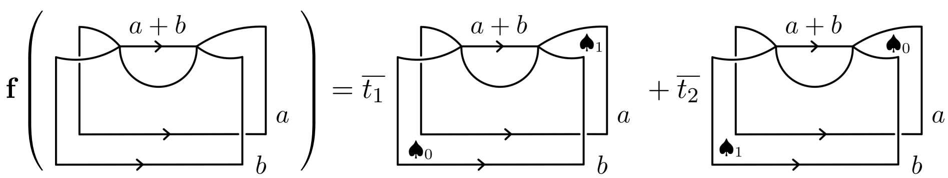

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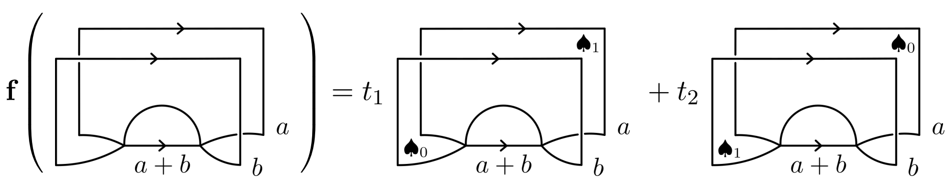

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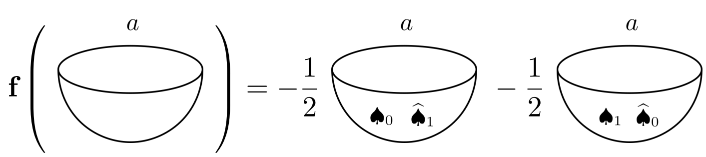

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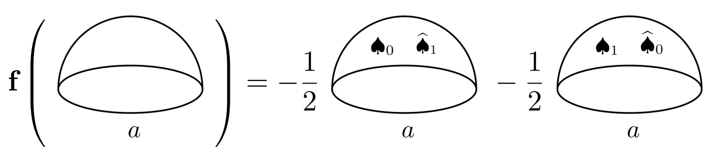

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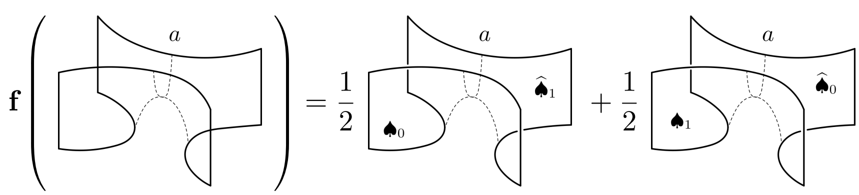


(49)

Next lemma and proposition are direct consequences of the previous section and proposition 3.3.

Lemma 3.7 Mapping $\mathtt{e}$ to $\mathbf{e}$, $\mathtt{h}$ to $\mathbf{h}$ and $\mathtt{f}$ to $\mathbf{f}$ for all $n \in \mathbb{N}_{-1}$ defines an action of $\mathfrak{sl}_2$ on the $\Bbbk$-module generated by foams in good position.

Proposition 3.8 Let F be a decorated closed foam, then for all $x \in \mathfrak{sl}_2$

$$\langle x \cdot F \rangle_N = x \cdot \langle F \rangle_N \tag{50}$$

3.4 $\mathfrak{gl}_N$-state space

In this section we build a functor between the category *Foam* (which refers to $Foam_{\mathbb{R}^2}$) and the category of projective, graded $\mathbb{Z}_N$-module. Let Γ be a web, and denote by $\mathcal{V}_N(\Gamma)$ the free $\mathbb{Z}_N$ -module generated by $\mathrm{Hom}_{Foam}(\varnothing,\Gamma)$. It is a graded by the degree of foams. Consider the $\mathbb{Z}_N$-bilinear form defined by:

$$\langle .,. \rangle_N \left| \begin{array}{ccc} \mathrm{Hom}_{Foam}(\varnothing,\Gamma) & \longrightarrow & \mathbb{Z}_N \\ (F,G) & \longmapsto & \langle F,G\rangle_N := \langle \bar{G}\circ F\rangle_N \end{array} \right. , \tag{51}$$

where $\overline{G}$ is the foam obtained by mirroring G along $\mathbb{R}^2 \times \left\{\frac{1}{2}\right\}$, so that $\overline{G}\circ F$ is a closed foam and $\langle \bar{G}\circ F\rangle_N$ is well defined. Moreover thanks to [RW18, Prop 2.18] we have $\langle \bar{G}\circ F\rangle_N \in \mathbb{Z}_N$. Let us denote:

$$\mathrm{Ker}(\langle .,. \rangle_N) := \bigcap_{G\in \mathrm{Hom}_{Foam}(\varnothing,\Gamma)} \mathrm{Ker}(\langle\ .\ ,G\rangle_N). \tag{52}$$

Now we can define the functor:

$$\mathcal{F}_N : \left| \begin{array}{ccc} Foam & \to & \mathbb{Z}_N\text{-Mod}_{gr} \\ \Gamma & \mapsto & \mathcal{V}_N(\Gamma)/\mathrm{Ker}(\langle .,. \rangle_N) \\ (F:\Gamma\to\Gamma') & \mapsto & (\psi_F : \mathcal{F}_N(\Gamma)\to\mathcal{F}_N(\Gamma')) \end{array} \right. \tag{53}$$

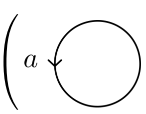
where ψ_F is induced by the following morphism:

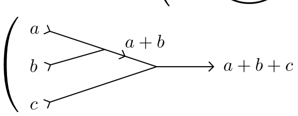
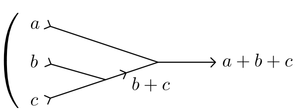
$$\tilde{\psi}_F \left| \begin{array}{ccc} \mathcal{V}_N(\Gamma) & \longrightarrow & \mathcal{V}_N(\Gamma') \\ G & \longmapsto & F\circ G \end{array} \right. \tag{54}$$

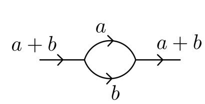
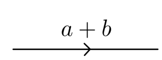
The categories *Foam* and $\mathbb{Z}_N$-Mod$_{gr}$ are both endowed with a monoidal structure. In *Foam* the tensor product is given by disjoint unions of webs and foams. The tensor product on $\mathbb{Z}_N$-Mod$_{gr}$ is given by the tensor product over $\mathbb{Z}_N$.

Proposition 3.9 The functor $\mathcal{F}_N$ is monoidal and satisfies the following local relations (and their mirror images):

$$\mathcal{F}_N(\varnothing) \simeq \mathbb{Z}_N \tag{55}$$

$\mathcal{F}_N\Big($  $\Big) \simeq \begin{bmatrix} N \\ a \end{bmatrix} \mathcal{F}_N(\varnothing)$ (56)

$\mathcal{F}_N\Big($  $\Big) \simeq \mathcal{F}_N\Big($  $\Big)$ (57)

$\mathcal{F}_N\Big($  $\Big) \simeq \begin{bmatrix} a+b \\ a \end{bmatrix} \mathcal{F}_N\Big($  $\Big)$ (58)

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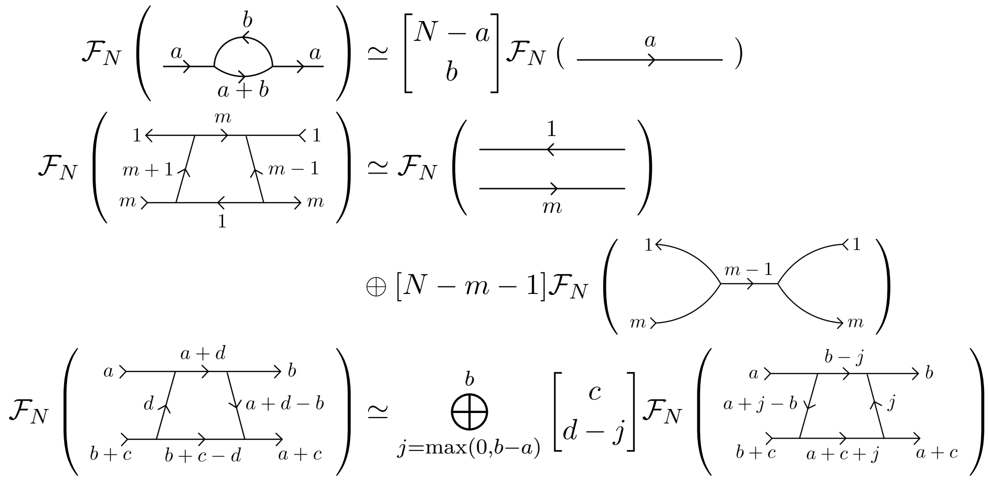


(60) (61) (62) (63)

These isomorphisms (except the first one) are realized as images of (linear combinations of) foams under $\mathcal{F}_N$.

Wu ([Wu13, Theorem 2.4]) proved that the relations given in Porposition3.9 are enough to reduce any web to empty set. Moreover $\mathcal{F}_N(\varnothing) \simeq \mathbb{Z}_N$ which is a finitely generated projective $\mathbb{Z}_N$-module. Since being projective and finitely-generated is preserved under finite direct sums and finite direct summands, the $\mathfrak{gl}_N$-state space of any web is a finitely generated projective $\mathbb{Z}_N$-module. Moreover thanks to Quillen Suslin's theorem, finitely generated projective $\mathbb{Z}_N$-modules are free.

Corollary 3.10 The functor $\mathcal{F}$ takes value in $\mathbb{Z}_N$-proj$_{gr}$, the category of finetely generated, graded, projective (and therefore free) $\mathbb{Z}_N$-modules.

For every web Γ, the action of $\mathfrak{W}^{\infty}_{-1}$ on foams gives an action of $\mathfrak{W}^{\infty}_{-1}$ on $\mathcal{F}_N(\Gamma)$. Indeed, this action is well defined because $\mathrm{Ker}(\langle .,. \rangle_N)$ is stable by the action of $\mathfrak{W}^{\infty}_{-1}$: Let F be a foam in $\mathrm{Ker}(\langle .,. \rangle_N)$ and $x \in \mathfrak{W}^{\infty}_{-1}$ then for all $G \in \mathrm{Hom}_{Foam}(\varnothing, \Gamma)$ we have:

$$\begin{aligned}\langle x \cdot F, G \rangle_N &= \langle G \circ (x \cdot F) \rangle_N \\ &= \langle x \cdot (G \circ F) \rangle_N - \langle (x \cdot G) \circ F) \rangle_N \\ &= x \cdot \langle G \circ F \rangle_N - \langle G' \circ F \rangle_N = 0.\end{aligned}$$

Last line is obtained by using the Proposition 3.6 and the fact that $G' := x \cdot G \in \mathrm{Hom}_{Foam}(\varnothing, \Gamma)$.

The action of $\mathfrak{W}^{\infty}_{-1}$ on foams is not compatible with the functor $\mathcal{F}_N$, indeed for all $x \in \mathfrak{W}^{\infty}_{-1}$ for all $F : \Gamma \to \Gamma'$ and for all $G \in \mathcal{F}_N(\Gamma)$:

$$\mathcal{F}_N(F)(x \cdot G) = F \circ (x \cdot G) = x \cdot (F \circ G) - (x \cdot F) \circ G = x \cdot \mathcal{F}_N(F)(G) - \mathcal{F}_N(x \cdot F)(G). \quad (64)$$

Due to the Leibniz rule, the $\mathbb{Z}_N$-linear map $\mathcal{F}_N(F)$ intertwines the action of $\mathfrak{W}^{\infty}_{-1}$ if and only if for all $x \in \mathfrak{W}^{\infty}_{-1}$, $\mathcal{F}_N(x \cdot F) = 0$. In this, case one says that $\mathcal{F}_N(F)$ is $\mathfrak{W}^{\infty}_{-1}$*−equivarient.* The previous statement can be apply to $\mathfrak{sl}_2$-action too.

4 Twist action

As seen above, $\mathfrak{W}^{\infty}_{-1}$-action and $\mathfrak{sl}_2$-action on foams and on states spaces are not always compatible. To circumvent this problem we introduce twist on state spaces.

4.1 About twists in general

This section is wrote in a general case and will be adapted on our two cases ($\mathfrak{sl}_2$ and $\mathfrak{W}_{-1}^{\infty}$) in the two next sections.

Let $\mathfrak{g}$ be a Lie algebra, A a commutative $\mathfrak{g}$-module algebra and M an $A\#\mathfrak{g}$-module, that a $\mathfrak{g}$-module structure with an A-module structure, which satisfies the following identity for all $g \in \mathfrak{g}, a \in A$ and $m \in M$:

$$g \cdot_{\mathfrak{g}} (a \cdot_A m) = (g \cdot a) \cdot_A m + a \cdot_A (g \cdot_{\mathfrak{g}} m).$$

A linear map $\tau : \mathfrak{g} \to A$ is *flat* if for all g_1, g_2 in $\mathfrak{g}$, one has:

$$\tau([g_1, g_2]) = g_1 \cdot \tau(g_2) + g_2 \cdot \tau(g_1).$$

If τ is flat, one can check that there is a new $A\#\mathfrak{g}$-module structure on the rank-one free module A it self define by:

$$g \cdot_{\mathfrak{g}^\tau} a := g \cdot_{\mathfrak{g}} a + \tau(g)a.$$

More generally, the following formula defines a new $\mathfrak{g}$-action on any $A\#\mathfrak{g}$-module M via:

$$g \cdot_{\mathfrak{g}^\tau} m := g \cdot_{\mathfrak{g}} m + \tau(g) \cdot_A m.$$

One check that the twisted action $\cdot_{\mathfrak{g}^\tau}$, similar as $\cdot_{\mathfrak{g}}$, is compatible with the A-action. In other words, this gives M a new $A\#\mathfrak{g}$-mod structure. If τ is flat, we denote by M^τ the module endowed with the action $\cdot_{\mathfrak{g}^\tau}$. One can readily see that, as $A\#\mathfrak{g}$-modules,

$$M^\tau = A^\tau \underset{A}{\otimes} M.$$

This means in particular that if $\tau, \sigma : \mathfrak{g} \to A$ are flat, then $(M^\tau)^\sigma \simeq M^{\tau+\sigma}$.

In our case, $\mathfrak{W}_{-1}^{\infty}$ (resp $\mathfrak{sl}_2$) is our Lie algebra and for each edge e with label a in a web Γ. We introduce the algebra:

$$D_e := \Bbbk[x_1, \dots, x_a, y_1, \dots, y_{N-a}]^{\mathfrak{S}_a \times \mathfrak{S}_{N-a}}$$

where $(x_i)_i$ are the variables used for the decorations of the facet associate to the edge e, and $(y_i)_i$ are the others variables. This algebra acts on the $\mathfrak{gl}_N$-state space $\mathcal{F}_N(\Gamma)$ associated to Γ by adding a decoration on the facet bounding to the edge e. Consequently the algebra

$$D_\Gamma := \bigotimes_{e \in E(\Gamma)} D_e$$

acts on $\mathcal{F}_N(\Gamma)$. For each edge e, the algebra D_e is endowed with a natural $\mathfrak{W}_{-1}^{\infty}$-module (resp $\mathfrak{sl}_2$-module) structure given by differential operators:

$$\mathbf{L}_n = -\sum_{i=0}^{a} x_i^{n+1} \frac{\partial}{\partial x_i} - \sum_{i=1}^{N-a} y_i^{n+1} \frac{\partial}{\partial y_i}.$$

and thus so is D_Γ.

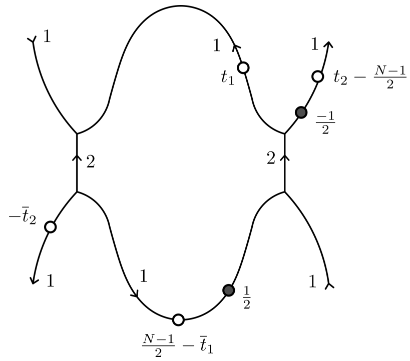


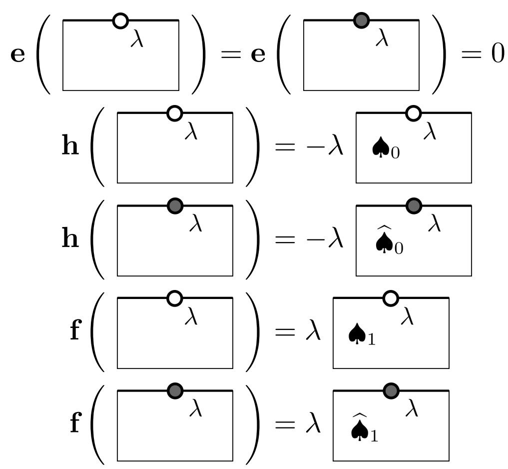
Figure 4: Example of a green-dotted web.

4.2 Green dots with $\mathfrak{sl}_2$-action

Definition 4.1 A *green-dotted* web is a web Γ endowed with a finite collection D of *green dots*, that are marked points with multiplicities (in $\Bbbk$) located in the interior of edges of Γ. These green dots are two types $\circ$ and $\bullet$. If a given edge carries several green dots of the same type, they may be replaced by one dot of that type on this edge with the sum of all multiplicities.

The Figure 4 shows an example of green-dotted web.

Let (Γ, D) be a green-dotted web. For each green dot d of multiplicity $\lambda \in \Bbbk$, define Γ_d to be the foam $\Gamma \times [0,1]$ with a twisted action of $\mathfrak{sl}_2$. Each green dot lives on an edge, hence each foam bounding Γ has a neighborhood of that green dots homeomorphic to $]0,1[\times[0,1[$. The twists induced by green dots is local and we decipt the modified $\mathfrak{sl}_2$-action on these neighborhoods.



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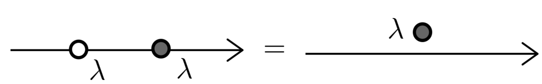
(69)

Recall that to act on a concrete foam, one uses the Leibniz rule, so that the twist induced by various green-dot on a given web add up.

It is convenient to introduce floating green dots on the plane: one can view them as being on edges of thickness 0. A floating hollow green dot $\circ$ do not alter the Lie algebra action since in this context $\spadesuit_0 = \spadesuit_1 = 0$. However a solid green dot $\bullet$ label by λ, twist the action of $\mathbf{h}$ by $-\lambda\widehat{\spadesuit}_0 = -\lambda N$ and the action of $\mathbf{f}$ by $\lambda\spadesuit_1 = \lambda P_1$. With this new convention, one has the following local relation:



(70)

Proposition 4.2 For any green-dotted web (Γ, D), twisting the actions of $\mathbf{e}, \mathbf{f}$ and $\mathbf{h}$ as above endows $\mathcal{F}_N(\Gamma)$ with an $\mathfrak{sl}_2$-module structure.

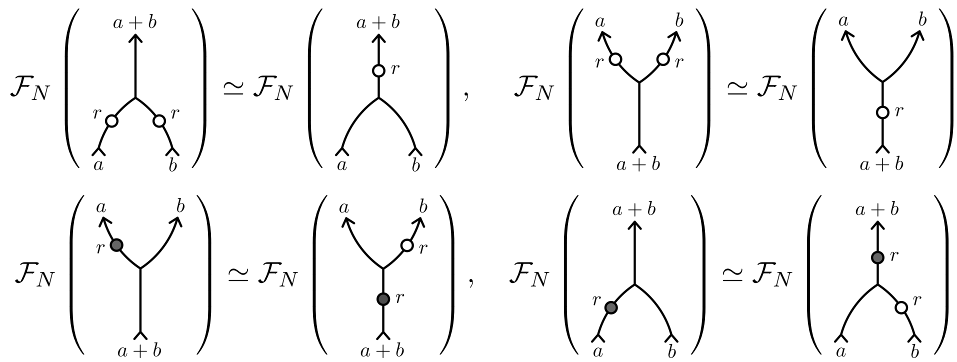
Proof. By additivity of twists, it is enought to prove for the case of a single edge e of thickness a contains a hollow dot labeled α and a solid dot labeled β. The verification of the flatness of $\tau : \mathfrak{sl}_2 \to D_e \subseteq D_\Gamma$ encoded by:

$$\tau(\mathbf{e}) = 0, \qquad \tau(\mathbf{h}) = -a\alpha - (N-a)\beta, \qquad \tau(\mathbf{f}) = \alpha\spadesuit_1 + \beta\widehat{\spadesuit}_1$$

is detailed on ([QRSW23, prop 3.13]). $\square$

After some computations we can allows green dots to migrate along a web in the following ways:

Lemma 4.3 The following $D_\Gamma \# \mathfrak{sl}_2$-modules are isomorphic:

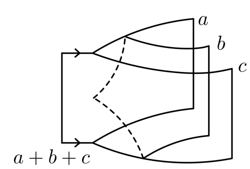


These manipulations of green dots, will be referred as *green dot migration.*

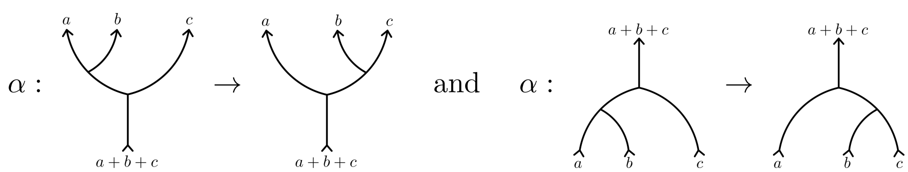
4.3 Useful morphism in the $\mathfrak{sl}_2$ case

In this section we introduce morphisms which play an important role in the link homology to be introduced in the section 5.

Lemma 4.4 The two different orientations of the foam

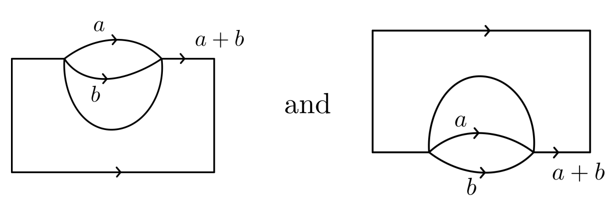


induce isomorphisms

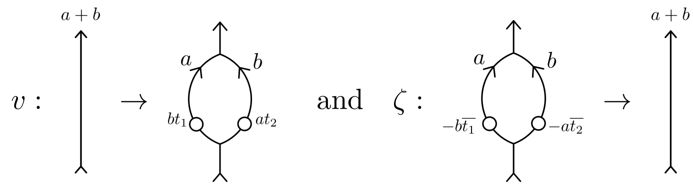


of $\mathfrak{sl}_2$-equivariant $\mathfrak{gl}_N$-state spaces associated with webs. Their inverses are also denoted by α.

Lemma 4.5 The foams

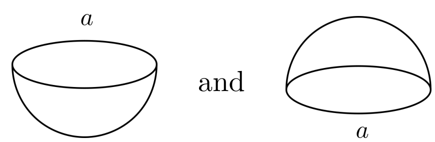


induce morphisms

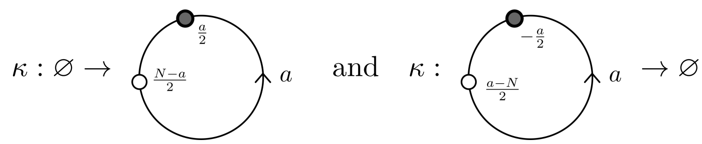


of $\mathfrak{sl}_2$-equivariant $\mathfrak{gl}_N$-state spaces associated with webs.

Lemma 4.6 The foams

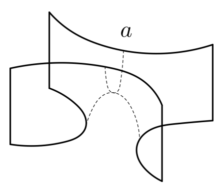


induce morphisms

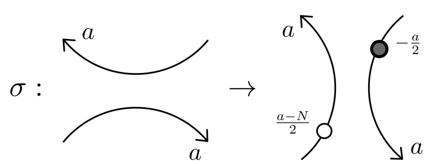


of $\mathfrak{sl}_2$-equivariant $\mathfrak{gl}_N$-state spaces associated with webs.

Lemma 4.7 The foam

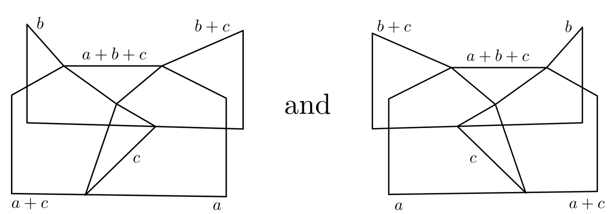


induces morphism

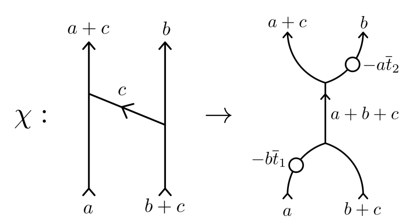


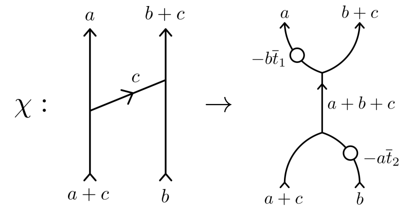
of $\mathfrak{sl}_2$-equivariant $\mathfrak{gl}_N$-state spaces associated with webs.

Lemma 4.8 The foams



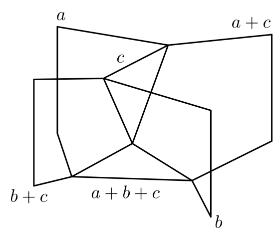
induce morphisms



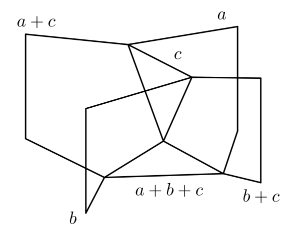


of $\mathfrak{sl}_2$-equivariant $\mathfrak{gl}_N$-state spaces associated with webs.

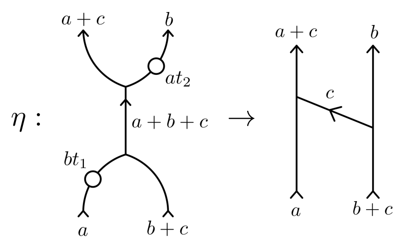
Lemma 4.9 The foams

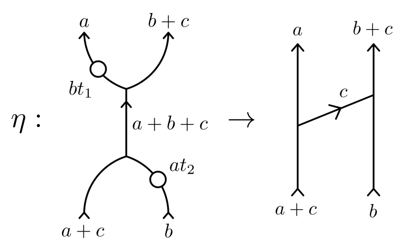


and



induce morphisms



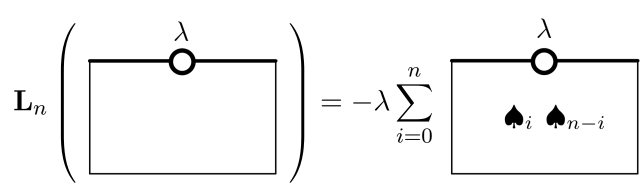


of $\mathfrak{sl}_2$-equivariant $\mathfrak{gl}_N$-state spaces associated with webs.

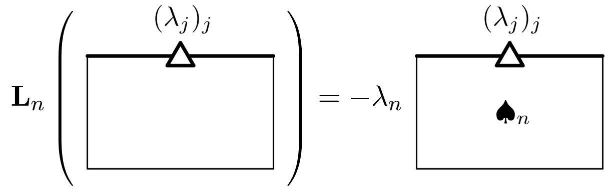
4.4 Green dots with $\mathfrak{W}_{-1}^{\infty}$-action

Definition 4.10 A *Witt green-dotted* web (also called green-dotted web) is a web Γ endowed with a finite collection D of *green dots*, that are marked points or triangles labeled by λ an element of $\Bbbk$ for points or $(\lambda_j)_j$ a Witt-sequence for triangles. These green dots are located in the interior of edges of Γ. There exist four types of green dots $\circ$, $\triangle$, $\blacktriangle$ and $\bullet$. If a given edge carries several green dots of the same type, they may be replaced by one dot of that type on this edge labeled by the sum of labels.

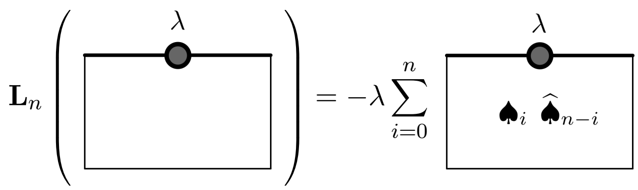
The twists induced by green dots is local and we depict the modified $\mathfrak{W}_{-1}^{\infty}$-action on these neighborhoods. Let λ an element of $\Bbbk$ and $(\lambda_j)_{j\in\mathbb{N}^{-1}}$ a Witt-sequence.



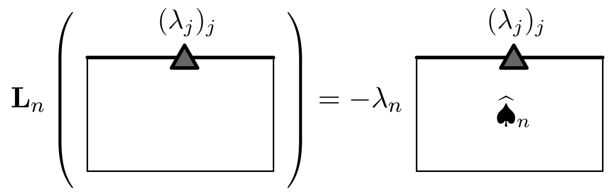
(71)



(72)



(73)



(74)

Recall that to act on a concrete foam, one uses the Leibniz rule, so that the twist induced by various green-dot on a given web add up.

Proposition 4.11 For any green-dotted web (Γ, D), twisting the action of $\mathbf{L}_n$ as above endows $\mathcal{F}_N(\Gamma)$ with an $\mathfrak{W}^{\infty}_{-1}$-module structure.

Proof. From a previous discussion, we know that if there are no green-dots, then the state space carries a $\mathfrak{W}^{\infty}_{-1}$-action.
By additivity of twists, it is enough to prove the proposition for the case that a single edge e of thickness a of contains a circular hollow dot labeled λ and a triangular hollow dot labeled $(\lambda_n)_{n\in\mathbb{N}_{-1}}$. The computation for a circular and a triangular solid dot is similar. Let $\tau \to D_\Gamma$ be the map encoded by these two green dots:

$$\tau(\mathbf{L}_n) = -\lambda \sum_{i=0}^{n} \spadesuit_i \spadesuit_{n-i} - \lambda_n \spadesuit_n.$$

We recall the action of $\mathbf{L}_n$ on $\spadesuit_i$:

$$L_n(\spadesuit_k) = -k\spadesuit_{k+n}.$$

Let us check if τ is flat.

$$\mathbf{L}_m \cdot \tau(\mathbf{L}_n) = \lambda \sum_{i=0}^{n} \left(i\spadesuit_{i+m}\spadesuit_{n-i} + (n-i)\spadesuit_i\spadesuit_{n+m-i}\right) + \lambda_n n \spadesuit_{n+m}$$

Thus,

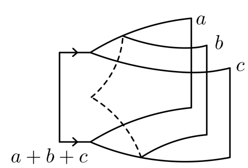
$$\begin{aligned}
\mathbf{L}_n \cdot \tau(\mathbf{L}_m) - \mathbf{L}_m \cdot \tau(\mathbf{L}_n) =&\lambda \sum_{i=0}^{m} (i\spadesuit_{i+n}\spadesuit_{m-i} + (m-i)\spadesuit_i\spadesuit_{n+m-i}) + \lambda_m m\spadesuit_{n+m} \\
&- \lambda \sum_{i=0}^{n} (i\spadesuit_{i+m}\spadesuit_{n-i} + (n-i)\spadesuit_i\spadesuit_{n+m-i}) - \lambda_n n\spadesuit_{n+m} \\
=&\lambda \sum_{i=0}^{n} (m-n)\spadesuit_i\spadesuit_{n+m-i} + \lambda \sum_{i=n+1}^{m} (m-i)\spadesuit_i\spadesuit_{n+m-i} + \lambda \sum_{i=n}^{m+n} (i-n)\spadesuit_i\spadesuit_{n+m-i} \\
&- \lambda \sum_{i=m}^{n+m} (i-m)\spadesuit_i\spadesuit_{n+m-i} + (m\lambda_m - n\lambda_n)\spadesuit_{n+m} \\
=&\lambda \sum_{i=0}^{n} (m-n)\spadesuit_i\spadesuit_{n+m-i} + \lambda \sum_{i=m}^{m+n} (m-n)\spadesuit_i\spadesuit_{n+m-i} + \lambda \sum_{i=n}^{m-1} (i-n)\spadesuit_i\spadesuit_{n+m-i} \\
&+ \lambda \sum_{i=n+1}^{m} (m-i)\spadesuit_i\spadesuit_{n+m-i} + (m-n)\lambda_{m+n}\spadesuit_{n+m} \\
=&(m-n)\lambda \sum_{i=0}^{m+n} \spadesuit_i\spadesuit_{m+n-i} + (m-n)\lambda_{n+m}\spadesuit_{n+m} \\
=&(n-m)\tau(\mathbf{L}_{m+n}).
\end{aligned}$$

□

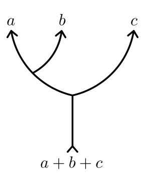
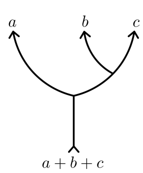
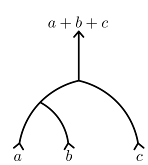
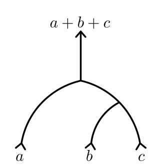
Remark 4.12 By choosing a triangular dot labeled by $(\frac{\lambda}{2}(j+1))_j$ one found the definition of green dots in the $\mathfrak{sl}_2$ case.

4.5 Useful morphism in the $\mathfrak{W}^{\infty}_{-1}$ case

Lemma 4.13 The two different orientations of the foam

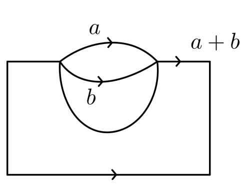


induce isomorphisms

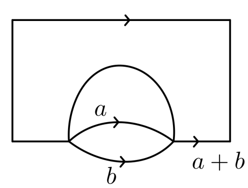
$\alpha :$  $\rightarrow$  and $\alpha :$  $\rightarrow$ 

of $\mathfrak{W}^{\infty}_{-1}$-equivariant $\mathfrak{gl}_N$-state spaces associated with webs. Their inverses are also denoted by α.

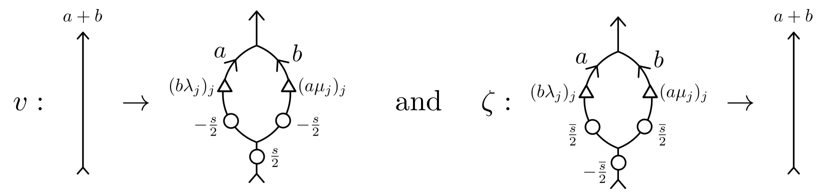
Lemma 4.14 The foams



and

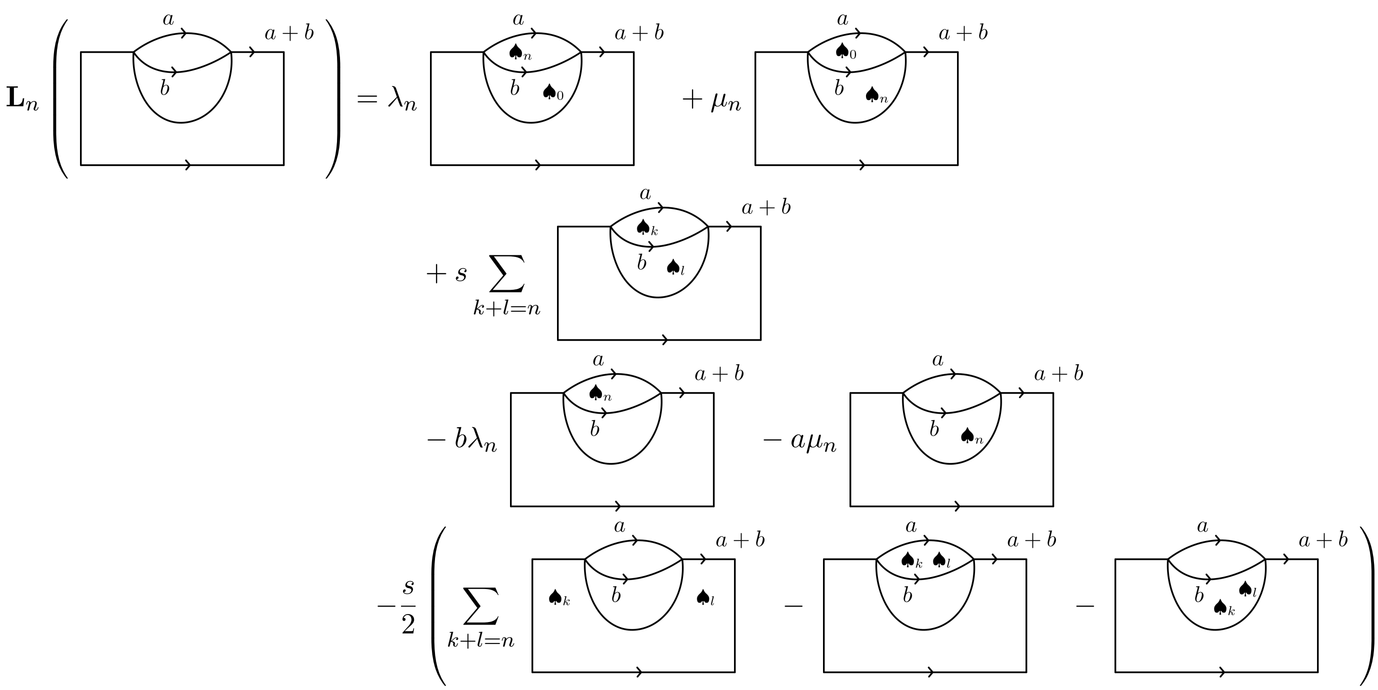


induce morphisms



of $\mathfrak{W}_{-1}^{\infty}$-equivariant $\mathfrak{gl}_N$-state spaces associated with webs.

Proof. We redact the proof for v. By using the Leibniz rule and the equations (21), (71) and (72) we have:

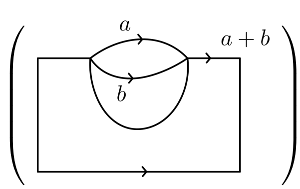


We denote by $\spadesuit_\cdot^a$ a decoration on the facet of thickness a. By definition of $\spadesuit_\cdot$ for all $i \in \mathbb{N}$ we have:

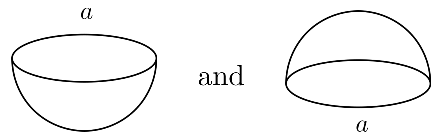
$$\spadesuit_i^{a+b} = \spadesuit_i^a + \spadesuit_i^b.$$

By using the equality, the computation between the parenthesis can be summarize by:

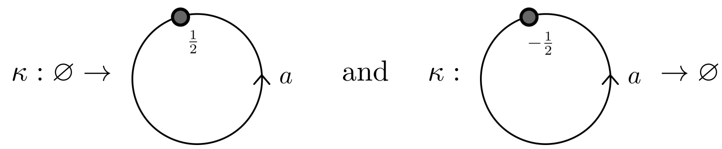
$$\begin{aligned}\sum_{k+l=n} \spadesuit_k^{a+b}\spadesuit_l^{a+b} - \spadesuit_k^a\spadesuit_l^a - \spadesuit_k^b\spadesuit_l^b &= \sum_{k+l=n} (\spadesuit_k^a + \spadesuit_k^b)(\spadesuit_l^a + \spadesuit_l^b) - \spadesuit_k^a\spadesuit_l^a - \spadesuit_k^b\spadesuit_l^b \\ &= \sum_{k+l=n} \spadesuit_k^a\spadesuit_l^b + \spadesuit_k^a\spadesuit_l^b \\ &= 2\sum_{k+l=n} \spadesuit_k^a\spadesuit_l^b\end{aligned}$$

Thus, we obtain that $\mathbf{L}_n$  $= 0$. □

Lemma 4.15 The foams

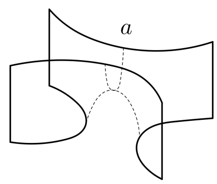


induce morphisms

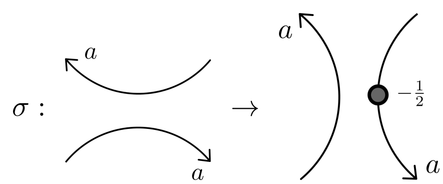


of $\mathfrak{W}^{\infty}_{-1}$-equivariant $\mathfrak{gl}_N$-state spaces associated with webs.

Lemma 4.16 The foam

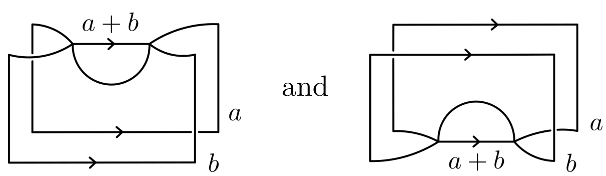


induces morphism

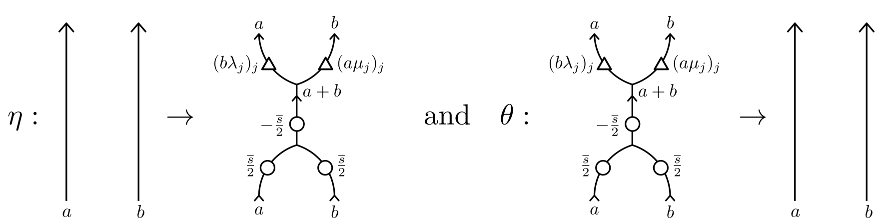


of $\mathfrak{W}^{\infty}_{-1}$-equivariant $\mathfrak{gl}_N$-state spaces associated with webs.

Lemma 4.17 The foams

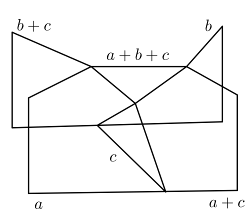


induce morphisms

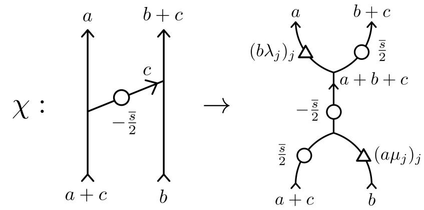


of $\mathfrak{W}^{\infty}_{-1}$-equivariant $\mathfrak{gl}_N$-state spaces associated with webs.

Lemma 4.18 The foam



induce morphism



of $\mathfrak{W}^{\infty}_{-1}$-equivariant $\mathfrak{gl}_N$-state spaces associated with webs.

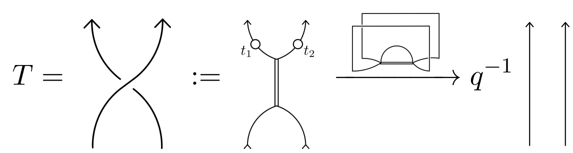
5 Link homology

In the section 3.4, we associated a $\mathbb{Z}_N$-module to any closed web. this construction leads to a definition to a definition of Khovanov-Rozansky $\mathfrak{gl}_N$-link homology. For more details, see [RW18]. We also saw in the previous section how $\mathfrak{sl}_2$ acts on the state space associated to a web. In this section we show how the $\mathfrak{sl}_2$-action extends to Khovanov-Rozansky homology. All proof of this section can be found in [QRSW23].

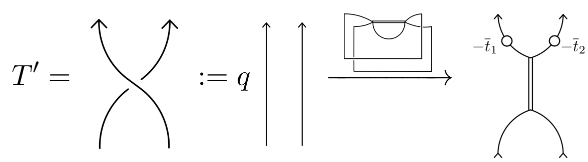
In the following section, edges will always have thickness 1 or 2 so we denote an edge of thickness 1 by (—) and an edge of thickness 2 by (═).

5.1 Link homology definition

As Khovanov homology ([Kau16]) we associate a hypercube shaped complex to any link by specifying locally a length 2 complex to any crossing. We define the following cohomologically graded braiding complexes:

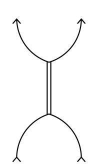


(75)



(76)

where in both complexes we assume that the terms



sit in cohomological degree 0. in this diagram $\mathcal{F}_N(\cdot)$ has been omitted to maintain readability.

For a link L, define $\mathrm{KR}^{\mathfrak{sl}_2}_N(L,R) := \mathrm{KR}^{\mathfrak{sl}_2}_{N,t_1,t_2}(L,R)$ to be the Khovanov-Rozansky $\mathfrak{gl}_N$-homology of L with coefficients in a ring R, equipped with the action of the Holf algebra $\mathcal{U}(\mathfrak{sl}_2)$. When the coefficient ring R is clear from context we will also write $\mathrm{KR}^{\mathfrak{sl}_2}_N(L)$ for simplicity. To prevent overloaded diagrams, we will mostly drop $\mathrm{KR}^{\mathfrak{sl}_2}_N(\cdot)$ around its.

The rest of the section will be devoted to proving that this link homology is invariant under Reidemeister moves.

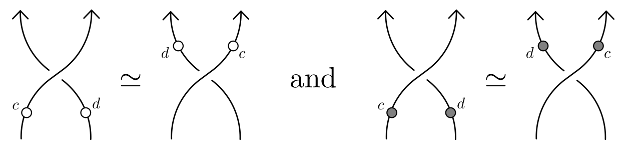
Theorem 5.1 The homology $\mathrm{KR}^{\mathfrak{sl}_2}_{N,t_1,t_2}(L,\mathbb{Z}_N)$ is an invariant of framed oriented links.

For a homology class α and an element $g \in \mathfrak{sl}_2$, we denote this action by $g \cdot \alpha$.

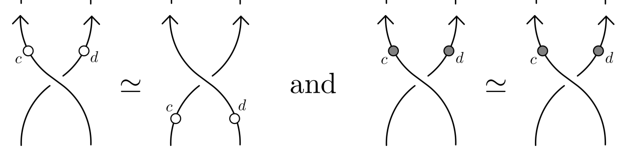
5.2 Reidemeister moves

We consider link diagram with green dots. These green dots should appear on any resolutions used to compute their homology. As we shall see, the homology of a link diagram with green dots does not depend on the precise location of green dots on a given component. Hence the green dot data can be thought as a framing. The proof of the next lemma come from ideas of [KR13] and is wrote in [QRSW23].

Lemma 5.2 Let $c, d \in \Bbbk$. Then there are isomorphisms of complexes in the relative homotopy category

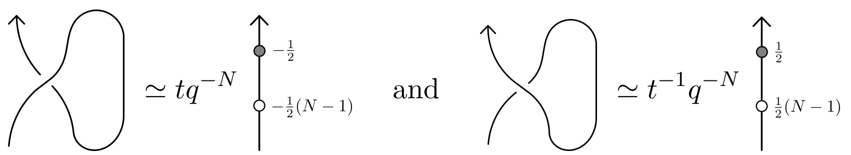


(77)



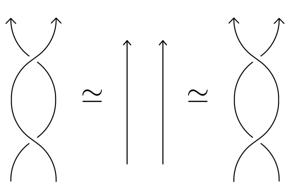
(78)

Proposition 5.3 (Reidemeister I)
There are isomorphisms in the relative homotopy category



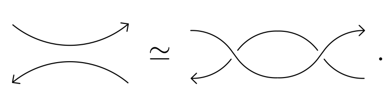
(79)

Proposition 5.4 (Reidemeister II)
There are isomorphismsin the relative homotopy category



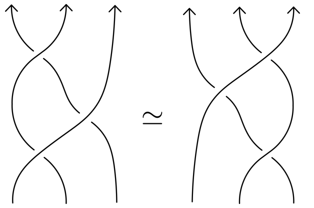
(80)

and,



(81)

Proposition 5.5 (Reidemeister III)
For $i = 1, \dots, n-1$, there are isomorphisms in the relative homotopy category



(82)

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