Schur polynomials notes

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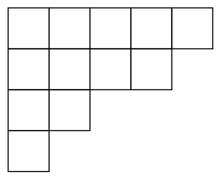
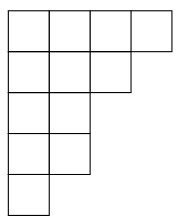
1 Partition

A *partition* is a finite $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $\lambda_i \geq \lambda_{i+1}$ and $\lambda_i \geq 0$.
The *weight* of λ is $|\lambda| = \lambda_1 + \ldots + \lambda_n = N$, λ is a partition of N.
The *length* of λ is $l(\lambda) = n$.

We can represent partitions as diagrams formally define as a set of square in position $(i, j) \in \mathbb{Z}^2$ (matrices convention) such that $1 \leq j \leq \lambda_i$. For example the diagram of the partition $\lambda = (5, 4, 2, 1)$ of 12 is:

$\lambda =$ 

The *conjugate* of a partition λ is the partition ${}^t\lambda$ whose diagram is the transpose of the diagram of lambda. For example, if $\lambda = (5, 4, 2, 1)$.

${}^t\lambda =$ 

Another notation for partitions which is occasionally useful is the following due to Frobenius. Suppose that the main diagonal of th diagram λ consists of r nodes (i, i). Let $\alpha_i = \lambda_i - i$ be the number of nodes in the ith row of λ to the right of (i, i), and let $\beta_i = {}^t\lambda_i - i$ be the number of nodes in the ith column of λ below (i, i), then we denote the partition λ by:

$$\lambda = (\alpha_1, \ldots, \alpha_r | \beta_1, \ldots, \beta_r) = (\alpha | \beta).$$

Proposition 1.1. Let λ be a partition and let $m \geq \lambda_1$, $n \geq^t \lambda_1$. Then the $m + n$ numbers

$$\lambda_i + n - i \quad (1 \leq i \leq n), \qquad n - 1 + j - {}^t\lambda_j \quad (1 \geq j \geq m)$$

are a permutation of $\{0, 1, \ldots, m + n - 1\}$.

Let λ and μ be two partitions, we define $\lambda + \mu$ to be the sum of the sequences λ and μ:

$$(\lambda + \mu)_i := \lambda_i + \mu_i.$$

Also we define $\lambda \cup \mu$ to be the partition whose parts are those of λ and μ, arranged in descending order. For example $\lambda = (3, 2, 1)$ and $\mu = (2, 1)$ then $\lambda + \mu = (5, 3, 1)$ and $\lambda \cup \mu = (3, 2, 2, 1, 1)$. We define $\lambda \times \mu$ to be the partition whose part are $(min(\lambda_i, \mu_j))$ for all $i \leq l(\lambda)$ and $j \leq l(\mu)$, arranged in decreasing order.
We have the following relations:

$$ {}^t(\lambda \cup \mu) = {}^t\lambda + {}^t\mu $$
$$ {}^t(\lambda \times \mu) = {}^t\lambda {}^t\mu $$

We can define an ordering $\mathscr{N}_n$ on $\mathscr{P}_n$, the set of partition of n, called the *natural (partial) ordering*:

$$\lambda >_{\mathscr{N}} \mu \Leftrightarrow \lambda_1 + \ldots + \lambda_i \geq \mu_1 + \ldots + \mu_i \quad \forall i \in [\![1, n]\!].$$

If $n \leq 6$ it is not a total ordering (ex: $(3, 1, 1, 1)$ and $(2, 2, 2)$).

2 Symmetric polynomials

2.1 The ring of symmetric polynomials Λ

$$m_\lambda(x_1, \ldots, x_n) = \sum_\alpha x^\alpha,$$

summed over all distinct permutations α of $\lambda = (\lambda_1, \ldots, \lambda_n)$. Therfore, $(m_\lambda)_\lambda$ is a $\mathbb{Z}$ basis of Λ.

2.2 Remarkable polynomials

For each $r \geq 0$ the rth *elementary symmetric polynomial* e_r, is the sum of all products of r distinct variables x_i, so that $e_0 = 1$ and

$$e_r = \sum_{i_1 < \ldots < i_r} x_{i_1} \ldots x_{i_r} = m_{(1^r)}$$

We have $\Lambda = \mathbb{Z}[e_0, e_1, \ldots]$. For each partition $\lambda = (\lambda_1, \lambda_2, \ldots,)$ define

$$e_\lambda := e_{\lambda_1} e_{\lambda_2} \ldots$$

Proposition 2.1. Let λ be a partition, ${}^t\lambda$ its conjugate. Then,

$$e_{{}^t\lambda} = m_\lambda + \sum_{\lambda >_{\mathscr{N}} \mu} a_{\lambda,\mu} m_\mu,$$

where the $a_{\lambda,\mu}$ are non-negative integers.

For each $r \geq 0$ the rth *complete symmetric polynomial* h_r, is the sum of all monomials of total degree r in the variables $x_1, x_2, \ldots$ so that

$$h_r = \sum_{|\lambda| = r} m_\lambda.$$

We have $\Lambda = \mathbb{Z}[h_1, h_2, \ldots]$

Proposition 2.2.

$$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0.$$

For each partition $\lambda = (\lambda_1, \lambda_2, \ldots,)$ define

$$h_\lambda := h_{\lambda_1} h_{\lambda_2} \ldots$$

$(h_\lambda)_\lambda$ is a $\mathbb{Z}$ basis of Λ.

2.3 Schur polynomials

Suppose to begin with that the number of variables is finite, say $x_1, \ldots, x_n$. Let $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ be a monomial, and consider the polynomial

$$a_\alpha = a_\alpha(x_1, \ldots, x_n) = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma)\ \sigma(x^\alpha).$$

We set that $a_\alpha = 0$ if α has more than n non empty (number of element in our set of variable) lines. Because a_α is skew-symmetric ie

$$\sigma(a_\alpha) = \varepsilon(\sigma) a_\alpha,$$

a_α is equal to zero unless $\alpha_1, \ldots, \alpha_n$ are all distinct. So $\alpha = \lambda + \delta$ where $\delta = (n-1, n-2, \ldots, 1, 0)$. Thus, we have,

$$a_\alpha = \det(x_i^{\lambda_j + n - j})_{1 \leq i,j \leq n}.$$

Because a_α is divisible by the Vandermonde, which is equal to a_δ, we can define the *Schur polynomial* associated to a partition λ:

$$s_\lambda = a_{\lambda+\delta}/a_\delta$$

that is a homogenous symmetric polynomial of degree $|\lambda|$. We set that $s_\lambda = 0$ if λ has more than n (number of element in our set of variable) non-empty lines.

We define [Mac98, I.4] the scalar product on Λ, by requiring that the $\mathbb{Z}$ bases (h_λ) and (m_λ) should be dual to each other:

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}.$$

Proposition 2.3. Each Schur polynomial s_λ can be expressed in the Λ-bases (e_k) and (h_k):

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \leq i,j \leq n}$$

where $n \geq l(\lambda)$.

$$s_\lambda = \det(e_{{}^t\lambda_i - i + j})_{1 \leq i,j \leq m}$$

where $m \geq l({}^t\lambda)$.

Example 2.4. Thank to the above relation we have:

$$s_{(1)^r} = e_r \quad \text{and} \quad s_{(k)} = h_k.$$

Proposition 2.5. The family of Schur polynomials $(s_\lambda)_\lambda$ is an orthogonal basis of Λ.

Let λ and μ be partitions, and define *the skew Schur polynomial* $s_{\lambda/\mu}$ thanks the coefficients:

$$\langle s_{\lambda/\mu}, s_\nu \rangle = \langle s_\lambda, s_\mu s_\nu \rangle =: c^\lambda_{\mu\nu},$$

thus, $s_{\lambda/\mu} = \sum_\nu \langle s_\lambda, s_\mu s_\nu \rangle s_\nu = \sum_\nu c^\lambda_{\mu\nu} s_\nu$. Furthermore we have the relation $s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda$. Also $c^\lambda_{\mu,\nu} = 0$ unless $|\lambda| = |\mu| + |\nu|$, so that $s_{\lambda/\mu}$ is homogenous of degree $|\lambda| - |\mu|$ and is zero if $|\lambda| < |\mu|$.

Proposition 2.6. The skew Schur polynomial $s_{\lambda/\mu}$ is zero unless $\mu \subset \lambda$, in which case it depend only on the skew diagram $\lambda - \mu$. If θ_i are the components of $\lambda - \mu$ then we have $s_{\lambda/\mu} = \prod s_{\theta_i}$. If the set of variable is finite $s_{\lambda/\mu}(x_1, \ldots, x_n) = 0$ unless $0 \leq {}^t\lambda_i - {}^t\mu_i \leq n$ for all $i \geq 1$.

Proposition 2.7. [Mac98, (5.9)] Let X and Y be two finite sets of independent variables. We have the following relation:

$$s_\lambda(X \sqcup Y) = \sum_\mu s_{\lambda/\mu}(X) s_\mu(Y) = \sum_{\mu,\nu} c^\lambda_{\mu,\nu} s_\mu(Y) s_\nu(X).$$

More generally,

$$s_{\lambda/\mu}(X \sqcup Y) = \sum_{\substack{\nu \\ \mu \subset \nu \subset \lambda}} s_{\lambda/\nu}(X) s_{\nu/\mu}(Y).$$

Proposition 2.8. [RW20, (A.6)] Let A, B and C be three sets of variables, and γ a Young diagram. Then the following identity holds:

$$\sum_{\alpha,\beta} c^\gamma_{\alpha\beta} (-1)^{|\alpha|} s_\alpha(A \cup C) s_{{}^t\beta}(B \cup C) = \sum_{i+j=n} c^\gamma_{\alpha\beta} (-1)^{|\alpha|} s_\alpha(A) s_{{}^t\beta}(B).$$

Proposition 2.9. If a and b are two non-negative integers, we have:

$$c^{\gamma}_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha \text{ is in } T(a,b) \text{ and } \beta = {}^t\hat{\alpha}, \\ 0 & \text{else} \end{cases}$$

Corollary 2.10. Let A, B and C be three sets of variables, and γ a Young diagram. Then the following identity holds:

$$\sum_{\alpha \in T(a,b)} (-1)^{|\alpha|} s_{\alpha}(A \cup C) s_{\hat{\alpha}}(B \cup C) = \sum_{\alpha \in T(a,b)} (-1)^{|\alpha|} s_{\alpha}(A) s_{\hat{\alpha}}(B).$$

Corollary 2.11. Let a and b two non-negative integers, and $Z = X \sqcup Y$ a set of variables. We have:

$$(-1)^{ab} s_{\rho(a,b)}(X) = \sum_{\alpha \in T(a,b)} (-1)^{|\alpha|} s_{\alpha}(Z) s_{\hat{\alpha}}(Y).$$

Corollary 2.12. We suppose that $Z = X \sqcup Y$, $|Z| = N$, and $X = \{x\}$. If $N \leq k$ are two positive integers, we have the following relation:

$$x^k = \sum_{i=0}^{N-1} (-1)^i h_{k-i}(Z) e_i(Y).$$

If $N > k$ we have:

$$x^k = \sum_{i=0}^{k} (-1)^i h_{k-i}(Z) e_i(Y).$$

(not the most general formulation)

Proof. By taking $a = k$, $b = 1$, $|X| = 1$, $|Z| = N \leq k$ in (2.11), we have:

$$(-1)^k s_{\rho(1,k)} = (-1)^k x^k,$$

and,

$$\begin{aligned}
\sum_{\alpha \in T(k,1)} (-1)^{|\alpha|} s_{\alpha}(Z) s_{\hat{\alpha}}(Y) &= \sum_{i=0}^{k} (-1)^i s_{(i)}(Z) s_{(1^{k-i})}(Y) \\
&= \sum_{i=k-N+1}^{k} (-1)^i s_{(i)}(Z) s_{(1^{k-i})}(Y) \\
&= \sum_{i=k-N+1}^{k} (-1)^i h_i(Z) e_{k-i}(Y) \\
&= \sum_{i=0}^{N-1} (-1)^{k-i} h_{k-i}(Z) e_i(Y).
\end{aligned}$$

□

Corollary 2.13. We suppose that $Z = X \sqcup Y$, $|Z| = N$, and $X = \{x\}$. We have the two following relations:

$$e_k(Y) = \sum_{i=0}^{N} (-1)^{k-i} e_i(Z) x^{k-i} \qquad \text{if } k \geq N$$

$$e_k(Y) = \sum_{i=0}^{k} (-1)^{k-i} e_i(Z) x^{k-i} \qquad \text{if } k < N$$

(not the most general formulation)

References

[Mac98] Ian Grant Macdonald. *Symmetric functions and Hall polynomials.* Oxford university press, 1998.

[RW20] Louis-Hadrien Robert and Emmanuel Wagner. A closed formula for the evaluation of foams. *Quantum Topol.*, 11(3):411–487, 2020. `doi:10.4171/qt/139`.